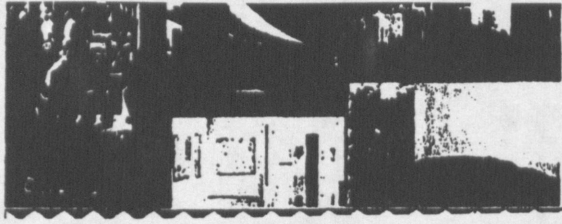

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Reliability of Complex Hierarchical Fault Tolerance Systems

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Abstract

A general model for describing, evaluation and control of most common reliability characteristics of complex hierarchical systems with fault tolerance units and different repair policies are proposed. Some algorithms for the steady state probabilities and the Laplace transforms of reliability functions are given. Particular examples illustrate the work of proposed algorithms.

1 Introduction and Motivation

Up-to-day complex technical systems are characterized by the following main properties:

- hierarchical structure;
- implemented system of the state control.

Hierarchy of structure means that the system consists of subsystems each of which is also divided on sub-subsystems etc. up to the lowest non-divided (elementary) level. We will refer to non-divisible part of system to as units. Usually hierarchy of structure leads to the property that the primary failures arise mainly at the lowest (elementary) level of the system, and gradually developing leads to the failure of blocks and subsystems of higher level, containing those elements. This means that the failures of the whole system are mainly not instantaneous, but are gradual, i.e. from the absolutely good state to full failure state the system goes through several intermediate states (fault stages). Such failures may change the state of the system and the quality of its operation, but do not necessarily lead to complete system failure. From abstract point of view these systems could be described as multi-state reliability systems. The wide review about this type of systems was given in [1], and present-day state of the subject one can find in [2]. Some type of multi-state hierarchical systems were considered in [3], [4].

Presence of implemented system of control (SoC) leads to the fact that the system became to be so called Fault Tolerance Systems (FTS). The SoC detects the faults and correct them itself or give a signal about necessity of the repair. Some times and costs (or reward lost) are needed for the repair. In the case if the system (subsystem or any block) is turns for the repair the further degradation impossible and it is supposed that after repair it became to be "as good as new" one. Different kind of preventive maintenance was considered by many authors (see Gertsbakh [5] and the references therein). This means that the reliability of the system is partially controllable. Different repair policies are possible after the gradual failure of some part of the system detection: the whole system, only failed element, or some structural part of the system can be repaired in this case. The case of the whole system repair was considered in [3], and the case of only unit repair was considered in [4].

Two main characteristics are common in the reliability studies: the life-time of the system, and its steady state characteristics under some assumptions about repair process. The ways to evaluate these characteristics depend on the approach to the following two aspects: probabilistic and structural. Probabilistic aspect deals with calculation of the system states probabilities, and uses them in reliability calculations. The structural aspect considers kind of direct evaluation of reliability characteristics for any given structure of a particular system.

The structure of system itself and its failure set are individual for each system and should be considered for each system (or class of systems) individually. Here we will not consider special structural properties of systems and focus on statistical properties of complex hierarchical systems reliability.

In this paper we propose a general approach to describe, model and evaluate the most common reliability characteristics of complex hierarchical systems with various types of gradual failures and different repair policies. Some special set of "failures states" F of component of the system cause its full failure. The repair policy is defined with special *repair function*. We propose the general equation for any repair function, and consider two special cases of this function, which lead to whole system repair or only to unit repair model. We deal here with only probabilistic aspects of modeling system reliability and focus on both of its common characteristics.

In the next section the model description is given. A formal mathematical model in the section 3 is proposed. Two special cases are considered in the next two sections. In the section 6 algorithms for calculation of the steady state and Lalace transform of time dependent probabilities are given, and some examples in the section 7 are considered.

2 A Model Description

Consider some *complex hierarchical multi-component system* subject to *gradual failures* of different types. Assume that the system is constructed from blocks and branches of several levels (see fig. 1). Each block and the following after branches and blocks forms a hierarchical subsystem of the same type as the main one. We will refer to the blocks of the last (lowest) level to as *units* and may be subjected to gradual failures of its own type. Some special combination F of units failures caused the whole system failure. We will denote by L the maximal level of units, and it is not necessary that any unit belongs to this level. Units of different levels are possible.

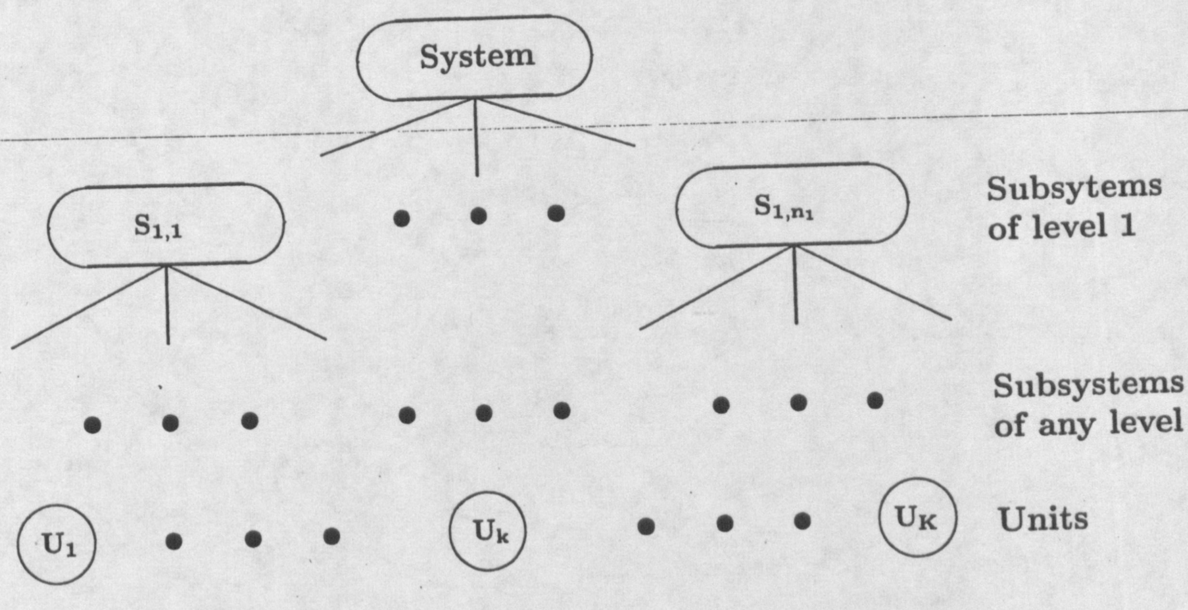

Fig. 1. A complex multi-level hierarchical system.

The reliability of the system is partially controllable. In case of a failure detection some part of the system accordingly to a given *repair function* $f(\mathbf{x})$ is renovated. This means that it is returned to its initial state (e.g. by replacing with a new one of the same type).

To specify the states space of the system and to define appropriate process describing its behavior let us introduce vector index $k = (i_1, i_2, \ldots, i_{L(k)})$ which determine each unit of the system as belonging to appropriate chain of blocks at any level with level of k-th unit denoted by $L(k)$. Denote also by $\mathcal{K}$ the set of these indices (and appropriate units) with $K = \#(\mathcal{K})$ number of units. Then the *states space* of the system can be represented as $E = \{\mathbf{x} = (x_k : k \in \mathcal{K})\}$, where for any $k \in \mathcal{K}$ the integer x_k represents the state of the k-th unit in sense of its reliability. It can take different values, depending on its type, $x_k \in \{0, 1, \ldots m_k\}$, where the exhausted state of k-th unit is denoted by m_k. Notice, that these numbers have no specific physical sense, but indicate only a possible level of gradual failure of the k-th unit. The value of $x_k = m_k$ means the full failure of k-th unit. Some special subset F of the system states space E specifies the failure of the system.

Due to implemented SoC each unit of the system does not fail immediately but follows through several stages, being so called *fault tolerance unit* (FTU). Let us consider the reliability model of some single FTU. Beginning from normal functioning (NF) state a FTU typically passes through the several stages: error detection (ED), damage assessment and confinement (DA), error recovery (ER), and fault treatment (FT) before it falls in the failure state (F). Nevertheless, due to unexpected failure it also could fails from any intermediate state (see fig. 2).

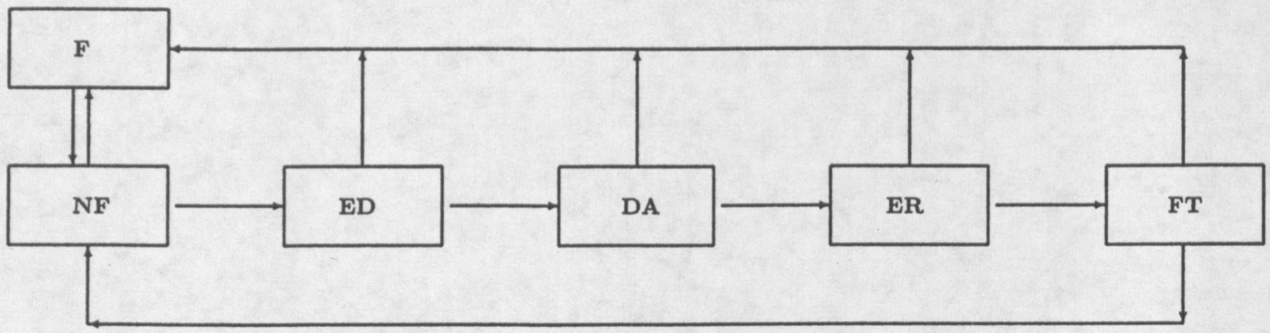

Fig. 2. State transition scheme for Single FTU Model.

The failures transfer the unit from one state to another, and these states are under control. It means that a gradual failure can be found and repaired. After any repair the unit is returned back to its initial state. Fault tolerance property means that the state space of the system can be divided into subsets of normal N, dangerous D and failure F states. In the next section a formal description of the model in framework of Discrete Markov Processes is given.

3 A Mathematical Model

To model the system functioning in accordance with a finite state Markov process we assume that the times of transition from one gradual level to another, as well as the repair of a failed unit have exponential distributions. The respective parameters may depend on the type of the unit $k \in \mathcal{K}$ and also on the entire system state $\mathbf{x}$. These general assumptions allow us to model the system reliability by using the multi-dimensional Markov process

$$\mathbf{X} = \{X_k(t) : k \in \mathcal{K}, \ t \geq 0\},$$

with set of states E, which should be specified for any particular system. Denote also by N the set of all normal system states, by D the set of dangerous states, and by F the set of the full system breakdown states. Moreover it is supposed that these subsets contain "boundary" sub-subsets Γ_{ND}, Γ_{NF} and Γ_{DF}, such that the transition to the states in D and F is possible only from the states of these subsets.

Additional assumption concerns the structure of transition intensities of such a process. The specific of the reliability models make it reasonable to suppose that the process can jump only in neighboring states (in the case that gradual failure arises), and some function $f = f_k(\mathbf{x})$ determined the states to which the process goes in the case of a fault in k-th unit is detected and appropriate repair is completed. The repair function f can be given by different way. The cases, when as repair result the whole system is renovated and only an unit is renovated were considered in [3], [4]. Another repair policies are also possible, for example, some subsystem of given level could be renovated as a repair result. This means that the transition intensities have the following form

$$a(\mathbf{x},\mathbf{y}) = \begin{cases} \alpha_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{x} + \mathbf{e}_k, \ \mathbf{x},\mathbf{y} \in N, \\ \lambda_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{x} + \mathbf{e}_k, \ \mathbf{x} \in \Gamma_N, \ \mathbf{y} \in D, \\ \beta_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{x} + \mathbf{e}_k, \ \mathbf{x},\mathbf{y} \in D, \\ \mu_k(\mathbf{x}) & \text{for } \mathbf{y} = f(\mathbf{x}), \ \mathbf{x} \in D, \\ \gamma_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{x} + \mathbf{e}_k, \ \mathbf{x} \in \Gamma_N, \ \mathbf{y} \in F, \\ \nu_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{x} + \mathbf{e}_k, \ \mathbf{x} \in \Gamma_N, \ \mathbf{y} \in F. \end{cases} \tag{1}$$

Here and later notation $\mathbf{e}_k$ means a unit vector with 1 at k-th position and zeros elsewhere, and $\alpha(\mathbf{x})$, $\beta(\mathbf{x})$, $\gamma(\mathbf{x})$, $\lambda(\mathbf{x})$, $\mu(\mathbf{x})$ and $\nu(\mathbf{x})$ denote the sums of appropriate intensities over all admissible in the state $\mathbf{x}$ indexes.

The graph of transition with appropriate intensities for typical states is shown at the figure 3.

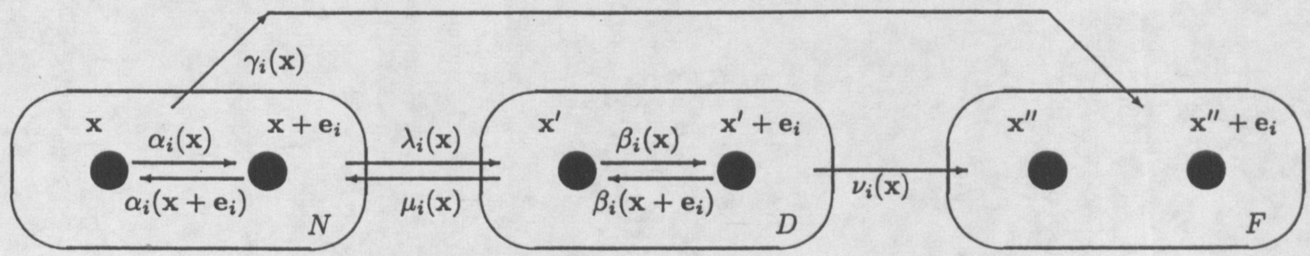

Fig. 3. Transition graph the model.

We will refer to the processes having these properties in possession as to *a Multi-State Reliability Process* (MSRP). The intensities $\lambda_k(\mathbf{x})$ and $\mu_k(\mathbf{x})$ will be called *failure* and *repair intensities* correspondingly. Different constrains to the state space E and/or the failure set F of this process and various assumption about dependence of transition intensities on the state give an opportunity to model a number of particular cases.

The Kolmogorov's system of differential equations for the time dependent probabilities of the process with transition intensities (1) and transition graph given at the figure 2 gets the form

$$\begin{aligned}
\frac{d\pi(0;t)}{dt} &= -\lambda(\mathbf{0})\pi(0;t) + \sum_{k\in\mathcal{K}} \mu_k(\mathbf{e}_k)\pi(\mathbf{e}_k;t) + \sum_{\mathbf{y}\in f^{-1}(\mathbf{0})\subset\mathcal{D}} \mu(\mathbf{y})\pi(\mathbf{y};t), \\
\frac{d\pi(\mathbf{x};t)}{dt} &= -(\alpha(\mathbf{x}) + \lambda(\mathbf{x}) + \gamma(\mathbf{x}))\pi(\mathbf{x};t) + \sum_{\mathbf{x}-\mathbf{e}_k\in N:\ x_k\neq 0} \alpha_k(\mathbf{x}-\mathbf{e}_k)\pi(\mathbf{x}-\mathbf{e}_k;t) + \\
&+ \sum_{\mathbf{y}\in f^{-1}(\mathbf{x})\subset\mathcal{D}} \mu(\mathbf{y})\pi(\mathbf{y};t), \qquad \mathbf{x}\in\mathcal{N}, \\
\frac{d\pi(\mathbf{x};t)}{dt} &= -(\beta(\mathbf{x}) + \mu(\mathbf{x}) + \nu(\mathbf{x}))\pi(\mathbf{x};t) + \sum_{\mathbf{x}-\mathbf{e}_k\in\Gamma_{ND}:\ x_i\neq 0} \lambda_k(\mathbf{x}-\mathbf{e}_k)\pi(\mathbf{x}-\mathbf{e}_k;t) + \\
&+ \sum_{\mathbf{x}-\mathbf{e}_k\in D:\ x_k\neq 0} \beta_k(\mathbf{x}-\mathbf{e}_k)\pi(\mathbf{x}-\mathbf{e}_k;t), \qquad \mathbf{x}\in\mathcal{D}, \\
\frac{d\pi(\mathbf{x};t)}{dt} &= \sum_{\mathbf{x}-\mathbf{e}_k\in\Gamma_{NF}} \gamma(\mathbf{x}-\mathbf{e}_k)\pi(\mathbf{x}-\mathbf{e}_k;t) + \sum_{\mathbf{x}-\mathbf{e}_i\in\Gamma_{DF}} \nu(\mathbf{x}-\mathbf{e}_k)\pi(\mathbf{x}-\mathbf{e}_k;t), \qquad \mathbf{x}\in F.
\end{aligned} \tag{2}$$

In general case this system of equations gives the possibility to investigate (at least numerically) both the stationary and the time dependent reliability characteristics of the system. Below we consider two cases, which admit close form and algorithmic solution.

To consider some special cases we define a partial order in E as follows:

$$\mathbf{x} < \mathbf{y}, \qquad \text{if } x_k \leq y_k \quad \text{for all } k\in\mathcal{K} \quad \text{and at least for one } x_k < y_k,$$

and we will use the following notations.

$\mathbf{x}_k(i)$ is the vector $\mathbf{x}$ with k-th component equals to i, i.e. $\mathbf{x}_k(i) = \mathbf{x} + (i - x_k)\mathbf{e}_k$;

Γ_r = is a hyper-plane with any of r-th components does not equal zero. To specify these components we will use

$\Gamma_r(j_1,\ldots,j_r)$ = $\{\mathbf{x}:\ x_{j_1}\neq 0,\ldots x_{j_r}\neq 0\}$ is a hyper-plane with $j_1,\ldots,j_r$ coordinates does not equal zero;

$$\Gamma(\mathbf{x}) \quad = \quad \{\mathbf{y}:\ y_j \neq 0 \text{ for all } j:\ x_j \neq 0\} \text{ is a hyper-plane, containing state } \mathbf{x}, \text{ and all other states } \mathbf{y} \text{ with the same non-zeroes component, as } x_j \neq 0 \text{ in } \mathbf{x};$$

$$p_r(\mathbf{x}) \quad = \quad p_r(\mathbf{0}=\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_l = \mathbf{x}) - \text{a monotone path from state } \mathbf{0} \text{ to state } \mathbf{x} \text{ through hyper-plane } \Gamma_r \text{ such that } \ \mathbf{0} < \mathbf{x}_1 < \ldots < \mathbf{x}_l = \mathbf{x} \quad \text{and } \ |\mathbf{x}_i - \mathbf{x}_{i-1}| = 1;$$

$$\alpha_i \quad = \quad \alpha(\mathbf{x}_{i-1}, \mathbf{x}_i) = \arg(\mathbf{x}_i - \mathbf{x}_{i-1}) - \text{label of unit, which failure leads to transition from state } \mathbf{x}_{i-1} \text{ to state } \mathbf{x}_i;$$

$$g(p(\mathbf{x})) \quad = \quad g(\mathbf{0}=\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_l = \mathbf{x}) = \prod_{1 \le i \le l} \frac{\lambda_\alpha(\mathbf{x}_{i-1})}{\gamma(\mathbf{x}_i)};$$

$$P_r(\mathbf{x}) \quad - \quad \text{the set of all monotone paths from state } \mathbf{0} \text{ to the state } \mathbf{x} \text{ through hyper-plane } \Gamma(\mathbf{x}) \text{ containing the state } \mathbf{x}\ ;$$

$$G_r(\mathbf{x}) \quad = \quad \sum_{p \in P_r(\mathbf{x})} g(p(\mathbf{x})) \quad \text{with} \quad G(\mathbf{0}) = 1. \tag{3}$$

Note that in this notations the set Γ_0 represents the point $\mathbf{0}$ and the set Γ_K represents the set of all "inner points" of the state space E. We will also omit the index r in the case if some statement takes place for any state $\mathbf{x}$. We begin with the model under whole system repair policy (SRP-model).

4 FTS under Whole System Repair Policy

To simplify the model we will not differ now the transition intensities inside the sets N, D, F and between these sets and preserve for them only notation λ and μ. For the system under whole SRP the repair function is $f_k(\mathbf{x}) = \mathbf{0}$ and therefore accordingly to the remark above the transition intensities take the form

$$a(\mathbf{x}, \mathbf{y}) = \begin{cases} \lambda_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{x} + \mathbf{e}_k, \\ \mu(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{0}, \end{cases} \tag{4}$$

and the structure of appropriate transition intensities graph is shown in the Fig. 3.

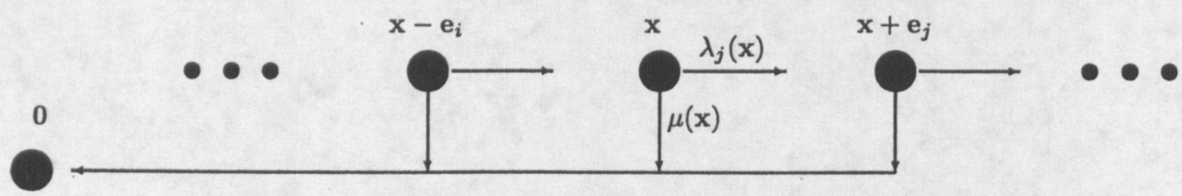

Fig. 3. Transition graph for whole system repair policy.

The Kolmogorov's system of differential equations for the time dependent probabilities of the process with transition intensities (4) gets the form

$$\begin{aligned} \frac{d\pi(\mathbf{0};t)}{dt} + \lambda(\mathbf{0})\pi(\mathbf{0};t) &= \sum_{\mathbf{x}\in E:\ \mathbf{x}\neq\mathbf{0}} \mu(\mathbf{x})\pi(\mathbf{x};t), \\ \frac{d\pi(\mathbf{x};t)}{dt} + \gamma(\mathbf{x})\pi(\mathbf{x};t) &= \sum_{k:\ x_k\neq 0} \lambda_k(\mathbf{x}-\mathbf{e}_k)\pi(\mathbf{x}-\mathbf{e}_k;t), \qquad \mathbf{x} \in E\setminus\{\mathbf{0}\}. \end{aligned} \tag{5}$$

This system can be used for calculation of steady state and time-dependent probabilities of the system.

The system of equations for stationary probabilities can be obtained by passing to limit when $t \to \infty$ in (5) and gets the form

$$\begin{aligned} \lambda(\mathbf{0})\pi(\mathbf{0}) &= \sum_{\mathbf{x}\in E,\ \mathbf{x}\neq\mathbf{0}} \mu(\mathbf{x})\pi(\mathbf{x}), \\ \gamma(\mathbf{x})\pi(\mathbf{x}) &= \sum_{k:\ x_k\neq 0} \lambda_k(\mathbf{x}-\mathbf{e}_k)\pi(\mathbf{x}-\mathbf{e}_k), \qquad \mathbf{x} \in E\setminus\{\mathbf{0}\}. \end{aligned} \tag{6}$$

Its solution can be represented in closed form. It is easy to see from the notation (3) that the definition of the function $G(\mathbf{x})$ provides the following recursive relation

$$G(\mathbf{x}) = \sum_{k:\ x_k>0} \frac{\lambda_k(\mathbf{x}-\mathbf{e}_k)}{\gamma(\mathbf{x})} G(\mathbf{x}-\mathbf{e}_k) \tag{7}$$

With this notation the following theorem was proved in the [3].

Theorem 1. *The steady state probabilities of the MSRP under whole system repair policy after failure detection have the form*

$$\pi_r(\mathbf{x}) = \left[\sum_{\mathbf{x}\in E} G(\mathbf{x})\right]^{-1} G_r(\mathbf{x}), \quad \textit{for any} \quad \mathbf{x}\in\Gamma_r\subset E. \qquad \square \tag{8}$$

From the theorem it follows

Corollary. *The failure probability π_F of the system equals to the sum of the steady state probabilities over all failure set,*

$$\pi_F = \sum_{\mathbf{x}\in F} \pi(\mathbf{x}). \qquad \square \tag{9}$$

The reliability function of the system coincides with the tail of the distribution for the time to first entrance of the process $X(t)$ into the failure set F. This distribution can be found by solving the system (5) with initial condition $\pi(\mathbf{0};0) = 1$, where any failure state is an absorbing state, $\lambda(\mathbf{x}) = \mu(\mathbf{x}) = 0$ for all $\mathbf{x}\in F$. In terms of Laplace transforms of time dependent probabilities

$$\tilde{\pi}(\mathbf{x};s) = \int_0^\infty e^{-st}\pi(\mathbf{x};t)dt \tag{10}$$

the system (5) is turned into the system of algebraic equations

$$\begin{aligned} (s+\lambda(\mathbf{0}))\tilde{\pi}(\mathbf{0};s) - \sum_{\mathbf{x}\in E:\ \mathbf{x}\neq\mathbf{0}} \mu(\mathbf{x})\tilde{\pi}(\mathbf{x};s) &= 1 \\ (s+\gamma(\mathbf{x}))\tilde{\pi}(\mathbf{x};s) - \sum_{k:\ x_k\neq 0} \lambda_k(\mathbf{x}-\mathbf{e}_k)\tilde{\pi}(\mathbf{x}-\mathbf{e}_k;s) &= 0 \quad \mathbf{x}\in E\setminus\mathbf{0}. \end{aligned} \tag{11}$$

To represent its solution denote $\tilde{\gamma}(\mathbf{x},s) = s+\gamma(\mathbf{x})$, and in the appropriate way change the notation for functions $g(\mathbf{x})$ and $G(\mathbf{x})$ in (3) by $\tilde{g}(\mathbf{x},s)$ and $\tilde{G}(\mathbf{x},s)$.

Theorem 2. ([3]) *The reliability function of the MSRP under whole system repair policy is*

$$R(t) = 1 - \pi_F(t), \tag{12}$$

where $\pi_F(t)$ is the distribution of the first entrance of the process $X(t)$ into the set F. It is given by

$$\pi_F(t) = \sum_{\mathbf{x}\in F} \pi(\mathbf{x};t), \tag{13}$$

and $\pi(\mathbf{x};t)$ has the Laplace transform

$$\tilde{\pi}(\mathbf{x};s) = \left[\tilde{\gamma}(\mathbf{0};s) - \sum_{\mathbf{x}\in E\setminus\mathbf{0}} \mu(\mathbf{x})\tilde{G}(\mathbf{x};s)\right]^{-1} \times \tilde{G}(\mathbf{x};s), \quad \mathbf{x}\in E. \qquad \square \tag{14}$$

5 FTS under Unit Repair Policy

Consider now the FTS under unit repair policy. The repair function for the case under consideration is $f_k(\mathbf{x}) = \mathbf{x}_k(0)$ and so in terms of the notations (3) the transition intensities (1) are

$$a(\mathbf{x},\mathbf{y}) = \begin{cases} \lambda_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{x}+\mathbf{e}_k, \\ \mu_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{x}_k(0), \end{cases} \tag{15}$$

and the transition graph has a form, shown on the Fig. 4.

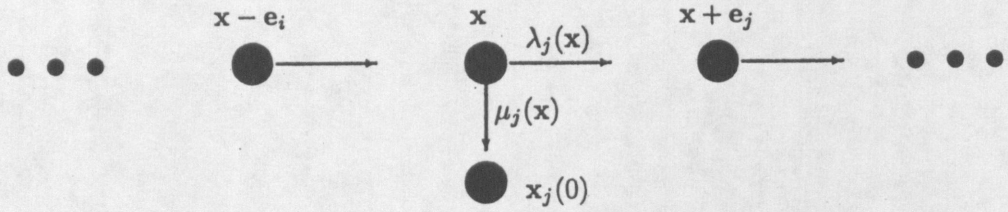

Fig. 4. Transition graph for the model under unit repair policy.

Now with the transitions intensities given by (15) the Kolmogorov's system of differential equations for the state dependent probabilities gets the form

$$\frac{d\pi(\mathbf{x};t)}{dt} + \gamma(\mathbf{x})\pi(\mathbf{x};t) = \sum_{k:\ x_k\neq 0} \lambda_k(\mathbf{x}-\mathbf{e}_k)\pi(\mathbf{x}-\mathbf{e}_k;t) + \sum_{k:\ x_k=0}\sum_{1\leq i\leq m_k} \mu_k(\mathbf{x}_k(i))\pi(\mathbf{x}_k(i);t), \qquad \mathbf{x}\in E. \quad (16)$$

Notice that this general form of equations takes place for the states at any hyper-plane, $\Gamma_r = \Gamma_r(j_1,\dots,j_r) = \{\mathbf{x}:\ x_{j_1}\neq 0,\dots,x_{j_r}\neq 0\}$.

The system of equations for stationary probabilities of the process can be obtained by eliminating derivatives (setting them all equal to 0) in left side of (16), i.e. by taking the limit in both sides when $t\to\infty$,

$$\gamma(\mathbf{x})\pi(\mathbf{x}) = \sum_{k:\ x_k\neq 0} \lambda_k(\mathbf{x}-\mathbf{e}_k)\pi(\mathbf{x}-\mathbf{e}_k) + \sum_{k:\ x_k=0}\sum_{1\leq i\leq m_k} \mu_k(\mathbf{x}_k(i))\pi(\mathbf{x}_k(i)), \qquad \mathbf{x}\in E. \quad (17)$$

Its solution can be represented in algorithmic form. To represent the solution of the equations (17) rewrite them in the form

$$\begin{aligned}\gamma(\mathbf{x})\pi_r(\mathbf{x}) &= \sum_{k:\ x_k>1} \lambda_k(\mathbf{x}-\mathbf{e}_k)\pi_r(\mathbf{x}-\mathbf{e}_k) + \sum_{k:\ x_k=1} \lambda_k(\mathbf{x}-\mathbf{e}_k)\pi_{r-1}(\mathbf{x}-\mathbf{e}_k) + \\ &+ \sum_{k:\ x_k=0}\sum_{1\leq i\leq m_k} \mu_k(\mathbf{x}_k(i))\pi_{r+1}(\mathbf{x}_k(i)), \qquad \mathbf{x}\in\Gamma_r,\end{aligned}$$

which indicates the hyper-plane to which the state $\mathbf{x}$ belongs. Unfortunately we can not represent the solution of this system in a closed form. In the following theorem we propose an iterative solution of these equations, in which the steady state probabilities from theorem 1 should be used as initial approximation. Remember that the steady state probabilities given by (8)could be considered for different hyper-planes.

Theorem 3. *The steady state probabilities for the MSRP under unit repair policy are a limit of successive approximations*

$$\pi_r(\mathbf{x}) = \lim_{n\to\infty}\pi_r^{(n)}(\mathbf{x}), \qquad \mathbf{x}\in\Gamma_r\subset E \quad (18)$$

given by the formulas

$$\begin{aligned}\pi_r^{(n+1)}(\mathbf{x}) &= \sum_{k:\ x_k>1} \frac{\lambda_k(\mathbf{x}-\mathbf{e}_k)}{\gamma(\mathbf{x})}\pi_r^{(n)}(\mathbf{x}-\mathbf{e}_k) + \sum_{k:\ x_k=1} \frac{\lambda_k(\mathbf{x}-\mathbf{e}_k)}{\gamma(\mathbf{x})}\pi_{r-1}^{(n)}(\mathbf{x}-\mathbf{e}_k) + \\ &+ \sum_{k:\ x_k=0}\sum_{1\leq i\leq m_k} \frac{\mu_k(\mathbf{x}_k(i))}{\gamma(\mathbf{x})}\pi_{r+1}^{(n)}(\mathbf{x}_k(i)), \qquad \mathbf{x}\in\Gamma_r, \quad (19)\end{aligned}$$

with the initial approximation, defined in the theorem 1,

$$\pi_r^{(0)}(\mathbf{x}) = \left[\sum_{\mathbf{x}\in E} G(\mathbf{x})\right]^{-1} G_r(\mathbf{x}), \quad \mathbf{x}\in\Gamma_r\subset E, \quad r=0,1,\dots,K.$$

Proof. The proof follows from the properties of irreducible Markov processes with finite states space. □

Corollary. *The failure probability π_F of the system under consideration equals to the sum of the steady state probabilities over all failure states, i.e.*

$$\pi_F = \sum_{x \in F} \pi(\mathbf{x}). \qquad \square \tag{20}$$

The recursive relations (19) provide a recursive algorithm for calculation of the steady state probabilities, and are discussed in the section 6.

The reliability function of the system can be found as the distribution of the time to first entrance of the process $X(t)$ into the failure set F. The distribution of the time to first entrance of $X(t)$ into the set F can be found by solving the system (16) with initial condition $\pi(0;0) = 1$, and all states $\mathbf{x} \in F$ are considered as absorbing states. The use of Laplace transform simplifies the solution of system (16), where $\lambda(\mathbf{x}) = \mu(\mathbf{x}) = 0$ is used for all failure states $\mathbf{x} \in F$. Applying the Laplace transform to system (16) with initial state at zero turns it into system of algebraic equations

$$\begin{aligned} (s + \lambda(\mathbf{0}))\tilde{\pi}(\mathbf{0}; s) - \sum_{k \in \mathcal{K}} \sum_{i=1}^{m_j} \mu_j(\mathbf{0}_j(i))\tilde{\pi}(\mathbf{0}_j(i); s) &= 1; \\ (s + \gamma(\mathbf{x}))\tilde{\pi}(\mathbf{x}; s) - \sum_{k\ x_k > 0} \lambda_k(\mathbf{x} - \mathbf{e}_k)\tilde{\pi}(\mathbf{x} - \mathbf{e}_k; s) - \sum_{k\ x_k = 0} \sum_{i=1}^{m_j} \mu_j(\mathbf{x}_k(i))\tilde{\pi}(\mathbf{x}_k(i); s) &= 0. \end{aligned} \tag{21}$$

The solution of this system of equations has the same structure as shown in Theorem 3. To represent this solution denote, as before, $\tilde{\gamma}(\mathbf{x}; s) = s + \gamma(\mathbf{x})$, and change notations for $G(\mathbf{x})$ by $\tilde{G}(\mathbf{x}; s)$ analogously to the previous case with tilde and additional argument "s", where also $\gamma(\mathbf{x})$ is changed to $\tilde{\gamma}(\mathbf{x}; s)$.

Theorem 4. *The reliability function of the MSRP has the form*

$$R(t) = 1 - \pi_F(t), \tag{22}$$

where $\pi_F(t)$ is the distribution of the first entrance of $X(t)$ into the set F. It is given by

$$\pi_F(t) = \sum_{\mathbf{x} \in F} \pi(\mathbf{x}; t),$$

and $\pi(\mathbf{x}; t)$ has the Laplace transform given by the successive approximations

$$\tilde{\pi}(\mathbf{x}; s) = \lim_{n \to \infty} \tilde{\pi}^{(n)}(\mathbf{x}; s), \tag{23}$$

with

$$\begin{aligned} \tilde{\pi}_r^{(n+1)}(\mathbf{x}; s) &= \sum_{k:\ x_k > 1} \frac{\lambda_k(\mathbf{x} - \mathbf{e}_k)}{\tilde{\gamma}(\mathbf{x})} \tilde{\pi}_r^{(n)}(\mathbf{x} - \mathbf{e}_k; s) + \sum_{k:\ x_k = 1} \frac{\lambda_k(\mathbf{x} - \mathbf{e}_k)}{\tilde{\gamma}(\mathbf{x})} \tilde{\pi}_{r+1}^{(n)}(\mathbf{x} - \mathbf{e}_k; s) + \\ &+ \sum_{k:\ x_k = 0} \sum_{1 \le i \le m_k} \frac{\mu_k(\mathbf{x}_k(i))}{\tilde{\gamma}(\mathbf{x})} \tilde{\pi}_{r-1}^{(n)}(\mathbf{x}_k(i); s), \qquad \mathbf{x} \in \Gamma_r, \end{aligned}$$

with the initial approximation, defined as

$$\tilde{\pi}(\mathbf{x}; s) = \left[\tilde{\gamma}(\mathbf{0}; s)\tilde{G}(\mathbf{0}; s) - \sum_{k \in \mathcal{K}} \sum_{1 \le i \le m_k} \mu_k(i\mathbf{e}_k)\tilde{G}(i\mathbf{e}_k; s) \right]^{-1} \tilde{G}(i\mathbf{x}; s), \qquad \mathbf{x} \in E.$$

The **proof** follows to the way used in the proof of Theorem 3 . $\square$

6 Algorithms

Recursive formulas (7) and (19) provide algorithms for calculation of the steady state and the Laplace transforms of the time dependent probabilities. Bellow we discuss algorithms for calculation of the steady state probabilities. The algorithms for calculation of the Laplace transforms of the time dependent probabilities follow similar procedure.

Algorithm 1. Calculation of the steady state probabilities for the system under system repair policy.

Begin. Input necessary information for the system.

Integers: $K,\ N$

Real: $\lambda_k(\mathbf{x}),\ \mu_k(\mathbf{x}),\ 1 \le k \le K,\ \mathbf{x} \in E.$

Define set F

Create Arrays: $g(\mathbf{x}),\ G(\mathbf{x})\ \mathbf{x} \in E.$

Step 1. For any monotone path from **0**-state to the state $\mathbf{x}$ of length $l = l(\mathbf{x}) = x_1 + \ldots + x_K$, beginning with $l = 0,\ G(\mathbf{0}) = 1$ calculate recursively the functions $G(\mathbf{x})$ accordingly to (7),

$$G_r(\mathbf{x}) = \sum_{k:\ x_k>0} \frac{\lambda_k(\mathbf{x}-\mathbf{e}_k)}{\gamma(\mathbf{x})} G_r(\mathbf{x}-\mathbf{e}_k)$$

for all $l = 1, 2, \ldots L = m_1 + \ldots + m_K$, and for all $\mathbf{x} \in \Gamma_r \subset E,\ r = 0, 1, \ldots, K.$

Step 2. Calculate the steady state probabilities $\pi(\mathbf{x})$ accordingly to formula (8)

$$\pi_r(\mathbf{x}) = \left[\sum_{x\in E} G(\mathbf{x})\right]^{-1} G_r(\mathbf{x}), \quad \mathbf{x} \in \Gamma_r \subset E,\ r = 0, 1, \ldots, K.$$

and the failure probability π_F according to formula (9)

$$\pi_F = \sum_{\mathbf{x}\in F} \pi(\mathbf{x}).$$

Step 3. Print results.

End.

For the system under unit repair policy the probabilities $\pi_r(\mathbf{x})$ depend on different values of r, so it could be calculated only recursively.

Algorithm 2. Calculation of the steady state probabilities for the system under unit repair policy.

Begin. Input necessary information about system.

Integers: $K,\ N;$

Real: $\lambda_k(\mathbf{x}),\ \mu_k(\mathbf{x}),\ 1 \le k \le K,\ \mathbf{x} \in E,\ \epsilon;$

Define sets $\Gamma_r(\mathbf{x}),\ P_r(\mathbf{x}),\ F;$

Create Arrays: $g(\mathbf{x}),\ G_r(\mathbf{x})\ 1 \le r \le K,\ \mathbf{x} \in E.$

Step 1. As in Algorithm 1 for any path from **0**-state to the state $\mathbf{x}$ of length $l = l(\mathbf{x}) = x_1 + \ldots + x_K$ beginning from $l = 0$ calculate recursively the functions $G_r(\mathbf{x})$ using formulas (7)

$$G_r(\mathbf{x}) = \sum_{k:\ x_k>0} \frac{\lambda_k(\mathbf{x}-\mathbf{e}_k)}{\gamma(\mathbf{x})} G_r(\mathbf{x}-\mathbf{e}_k), \qquad \mathbf{x} \in \Gamma_r,\ r = 0, 1, \ldots K$$

for all $l = 1, 2, \ldots L = m_1 + \ldots + m_K$, and for all $\mathbf{x} \in \Gamma_r \subset E,\ r = 0, 1, \ldots, K.$

Step 2. Calculate the initial approximation to the steady state probabilities $\pi(\mathbf{x})$ accordingly to formula (8)

$$\pi_r^{(0)}(\mathbf{x}) = \left[\sum_{x\in E} G(\mathbf{x})\right]^{-1} G_r(\mathbf{x}), \quad \mathbf{x} \in \Gamma_r \subset E,\ r = 0, 1, \ldots, K.$$

Step 3. For all hyper planes of any order r, beginning from $n = 0$, and given $\pi_r^{(0)}(\mathbf{x})$ recursively calculate the probabilities $\pi_r^{(n)}(\mathbf{x})$ using formula (19):

$$\begin{aligned}\pi_r^{(n+1)}(\mathbf{x}) \quad &= \quad \sum_{k:\ x_k>1} \frac{\lambda_k(\mathbf{x}-\mathbf{e}_k)}{\gamma(\mathbf{x})} \pi_r^{(n)}(\mathbf{x}-\mathbf{e}_k) + \sum_{k:\ x_k=1} \frac{\lambda_k(\mathbf{x}-\mathbf{e}_k)}{\gamma(\mathbf{x})} \pi_{r-1}^{(n)}(\mathbf{x}-\mathbf{e}_k) + \\ &+ \quad \sum_{k:\ x_k=0} \sum_{1\le i\le m_j} \frac{\mu_j(\mathbf{x}_j(i))}{\gamma(\mathbf{x})} \pi_{r+1}^{(n)}(\mathbf{x}_j(i))\end{aligned}$$

for all $\mathbf{x} \in \Gamma_r \subset E$ and for all $r = 0, 1, 2, \ldots K.$

Step 4. Repeat the step 3 up to condition

$$\max_{0\le r\le K}\max_{\mathbf{x}\in\Gamma_r}|G_r^{(n+1)}(\mathbf{x})-G_r^{(n)}(\mathbf{x})|<\epsilon$$

holds. Put $\pi_r^{(n)}(\mathbf{x})=\pi_r(\mathbf{x})$.

Step 5. Find the failure probability π_F according to (20):

$$\pi_F=\sum_{x\in F}\pi(\mathbf{x}).$$

Step 6. Print results.

End.

Some specific additional steps should be introduced and new formula should be developed in order to take into account the special structural restrictions for the failure set, repair policy and other structural properties of the system. Any additional performance characteristics of the system reliability can be calculated using steady state probabilities and the specific structural properties of the system.

7 Examples

Consider a homogeneous hierarchical system with K units, each of which may pass through only two stages of reliability, i.e. $m=1$, and only one failed unit coulf be repaired simulteneously. This means, that the failure and repair intensities are

$$\lambda_k(\mathbf{x})=\lambda,\quad \mu_k(\mathbf{x})=\mu.$$

Remember that Γ_r denotes the hyper plane with r components of vector $\mathbf{x}$ does not equal zero. Therefore for any $\mathbf{x}\in\Gamma_r$ it holds

$$\lambda(\mathbf{x})=(K-r)\lambda\quad\text{and}\quad\gamma(\mathbf{x})=(K-r)\lambda+r\mu.$$

7.1 Whole System Repair Policy Model

For the model with whole system repair policy due to homogeneity and one failure stage we have $g(0)=g_0=1$, and for any $\mathbf{x}\in\Gamma_r$

$$g(\mathbf{x})=g_r=\prod_{1\le i\le r}\frac{\lambda}{(K-i)\lambda+i\mu},\qquad r=1,\ldots,K.$$

Notice that the number of distinct paths from state $\mathbf{0}$ to any state $\mathbf{x}\in\Gamma_r$ with r components of vector $\mathbf{x}$ equal one is $r!$, $\#[\Gamma_r]=r!$. Therefore, for any $\mathbf{x}\in\Gamma_r$ the function $G(\mathbf{x})$ can be presented in the form

$$G(\mathbf{x})=r!\prod_{1\le i\le r}\frac{\lambda}{(K-i)\lambda+i\mu}=\prod_{1\le i\le r}\rho_i,\quad r=1,\ldots,K$$

with $G(\mathbf{0})=\mathbf{1}$ and the notation

$$\rho_i=\frac{i\lambda}{(K-i)\lambda+i\mu}.$$

Since the number of states in any hyper plane Γ_r equals $\binom{K}{r}$, the normalizing constant C gets the form

$$C=\left[\sum_{0\le r\le K}\binom{K}{r}\prod_{1\le i\le r}\rho_i\right]^{-1}=\left[1+\rho_1\left(\binom{K}{1}+\rho_2\left(\binom{K}{2}+\ldots+\rho_K\binom{K}{K}\cdots\right)\right)\right]^{-1}.$$

Therefore, for any $\mathbf{x}\in\Gamma_r$ holds

$$\pi(\mathbf{x})=\pi_r=C\prod_{1\le i\le r}\rho_i.$$

If only the failure of all units leads to the system failure, we get

$$\pi_F=\pi(\mathbf{1})=\pi_K=C\prod_{1\le i\le K}\rho_i.$$

If failure of any unit causes the system failure, then

$$\pi_F = 1 - \pi(\mathbf{0}) = 1 - \pi_0 = 1 - C.$$

If the system fails only when L units, or more are failed then probability of failure is

$$\pi_F = \sum_{r \geq L} \pi_r = 1 - C \sum_{0 \leq r \leq L-1} \binom{K}{r} \prod_{1 \leq i \leq r} \rho_i.$$

Remark that due to homogeneity the process $\mathbf{X}$ admit states aggregation, and the same result could be obtained with the state aggregation method.

7.2 The Unit Repair Policy Model

To calculate the steady state probabilities for the model under only a failed unit repair policy we use the state aggregation method. Due to homogeneouity the states in any hyper-plane Γ_r admit aggregation. After aggregation of the states in any hyper-plane Γ_r the process $\mathbf{X}$ take the form of birth and death process and the equations (17) take the form

$$\begin{aligned} K\lambda\hat{\pi}_0 &= K\mu\hat{\pi}_1, \\ ((K-r)\lambda + r\mu)\hat{\pi}_r &= (K-(r-1))\lambda\hat{\pi}_{r-1} + (r+1)\mu\hat{\pi}_{r+1} \qquad (r = 1, \ldots, K-1), \\ K\mu\hat{\pi}_K &= \lambda\hat{\pi}_{K-1}, \end{aligned}$$

where $\hat{\pi}_r$ denotes the steady state probabilities of the aggregated states,

$$\hat{\pi}_r = \mathbf{P}\{\mathbf{X} \in \Gamma_r\}.$$

From these equations one can find that

$$\hat{\pi}_r = \binom{K}{r} \frac{\rho^r}{(1+\rho)^K},$$

where the new parameter $\rho = \lambda/\mu$ is used.

If only the failure of all units leads to the system failure, we get

$$\pi_F = \pi(\mathbf{1}) = \pi_K = \hat{\pi}_K = \frac{\rho^K}{(1+\rho)^K}.$$

If failure of any unit causes the system failure, then

$$\pi_F = 1 - \pi(\mathbf{0}) = 1 - \pi_0 = 1 - \hat{\pi}_0 = 1 - \frac{1}{(1+\rho)^K}.$$

7.3 A Numerical Illustration

To illustrate the model under the both repair policies we consider the two blocks two levels system with only one stages failure of each unit. In this system $K = 4$ and $m = 1$. For this case the hyper planes Γ_r contain the following states:

$$\begin{aligned} \Gamma_0 &= \{(0,0,0,0)\}; \\ \Gamma_1 &= \{(1,0,0,0),\ (0,1,0,0),\ (0,0,1,0),\ (0,0,0,1,)\}; \\ \Gamma_2 &= \{(0,0,1,1),\ (1,0,0,1),\ (0,1,1,0),\ (1,0,0,1),\ (1,0,1,0),\ (1,1,0,0)\}; \\ \Gamma_3 &= \{(1,1,1,0),\ (1,1,0,1),\ (1,0,1,1),\ (0,1,1,1,)\}; \\ \Gamma_4 &= \{(1,1,1,1)\}. \end{aligned}$$

We investigate the failure probabilities for both models as functions of parameter ρ. These functions are:

- $\pi_F^{(1)}$ the failure of the system under whole systen repair policy if the failure of all units lead to the system failure;
- $\pi_F^{(2)}$ the failure of the system under whole systen repair policy if the failure of any one unit causes the system failure;

- $\pi_F^{(3)}$ the failure of the system under only failed unit repair policy if the failure of all units lead to the system failure;
- $\pi_F^{(4)}$ the failure of the system under only failed unit repair policy if the failure of any one unit causes the system failure.

The expression of these functions are:

$$\begin{aligned}
\pi_F^{(1)} &= \frac{6C}{(3+\rho)(2+2\rho)(1+3\rho)\rho};\\
\pi_F^{(2)} &= 1-C, \qquad \text{with}\\
C &= \frac{1}{6}(3+\rho)(2+2\rho)(1+3\rho)\rho;\\
\pi_F^{(3)} &= \frac{\rho^4}{(1+\rho)^4};\\
\pi_F^{(4)} &= 1-\frac{1}{(1+\rho)^4}.
\end{aligned}$$

In the graphs bellow the failure probabilities for both models as functions of parameter ρ are shown.

8 Conclusion

Some general solutions for models of system reliability with gradual failures are derived and related algorithms are proposed. Special computer tools are needed for real reliability system elaboration. The algorithms show that , in general, the dimension of the problem grows with the complexity of the system. Nevertheless some simple examples indicate that these algorithms can be successfully used for reasonable problem solution.

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