Boyan DIMITROV
David GREEN
Vladimir RYKOV
Peter STANCHEV

ON THE FAIR SHARE OF THE RELIABILITY OF AN ENTITY BETWEEN ITS COMPONENTS

The problem of the reliability of an entity sharing between their components in order to maximize its lifetime is considered. Some algorithms generating solutions to the problem is presented along with numerical examples for the problem.

Keywords: *optimization, reliability systems, lifetime maximization*

1. Introduction

Most of real engineering entities (products, goods) consists of various components and usually has a complex hierarchical structure. Their components have different reliability, cost and other characteristics. The entity's reliability is usually determined by the reliability of the weakest component among them. Thus, it then becomes necessity to investigate how to construct the system in order to uniformly maximize its reliability. As mathematical tools for investigation of these kinds of systems the multi-state system reliability models could be used.

The problem of multi-state system reliability investigation was considered by different authors, and one can find the bibliography in Lisnianski and Levitin (2003). Some special approach to this problem for complex hierarchical systems also was developed in several papers (Dimitrov et al. 2004, Dimitrov et al. 2002, Dimitrov and Rykov 2002). Some problems of reliability control were considered in Rykov and Efrosinin (2004). In (Ermolaev and Rykov 2000) the problem of optimal reservation with different types of equipment was considered. In this paper we consider the problem of a system reliability sharing between its components with respect to the system lifetime maximization.

2. Problems settings

Consider an entity consisting of m components having lifetimes T_i with cumulative probability distribution functions (c.d.f.) $F_i(t)$ $(i = 1,2,\ldots,m)$. Denote by $R_i(t) = 1-F_i(t)$ the reliability functions of the i–th component. Suppose that in accordance with consumer requirements the entity should be given reliability function of level at least $r = 1-\alpha$. It means that the probability for the entity to fail should be only α, or less.

The usual opinion that the equally reliable components provide the best reliability for the system is not really true. To explain this fact let us consider the following examples.

2.1. Consequence system

For a system with consequently connected components, each of which has an exponential lifetime distribution with parameters λ_i, $(i=\overline{1,m})$, the reliability function of the system equals (Gertsbakh 2000)

$$R(t)=\exp\left\{-\left(\sum_{1\le i\le m}\lambda_i\right)t\right\}\equiv e^{-\Lambda t}\quad \text{with}\quad \Lambda=\sum_{1\le i\le m}\lambda_i$$

The reliability level $r = 1-\alpha$ will be provided up to time $t_{1-\alpha}=-\frac{\ln(1-\alpha)}{\Lambda}\approx\frac{\alpha}{\Lambda}$. The reliability level of i-th component of the system for this time will be equal

$$R_i(t_{1-\alpha})=e^{-\lambda_i t_{1-\alpha}}=\exp\left\{\frac{\lambda_i}{\Lambda}\ln(1-\alpha)\right\}=(1-\alpha)^{\frac{\lambda_i}{\Lambda}}\approx 1-\alpha_i.$$

Note that the equally reliable sharing of the probability between subsystems when reliability level for each component equals $(1-\alpha)^{1/m}$ provides the guaranteed lifetime for i-th component only

$$t_{i,1-\alpha}=-\frac{1}{m\lambda_i}\ln(1-\alpha)\approx\frac{\alpha}{m\lambda_i}.$$

Thus, the $(1-\alpha)$ guaranteed lifetime level for a system will be equal

$$t_{1-\alpha}=\min_{1\le i\le m}t_{i,1-\alpha}=-\frac{\ln(1-\alpha)}{m}\frac{1}{\max\lambda_i}\approx\frac{\alpha}{m\max\lambda_i}$$

If we consider some simple case of the system with only two components with parameters $\lambda_1 = 0.1$ and $\lambda_2 = 0.01$ then the equally reliable sharing of the system reliability provides $1-\alpha$ guaranteed lifetime equals $t_{1-\alpha} = \min[5\alpha,50\alpha] = 5\alpha$, while an optimal sharing provide the time $t^*_{1-\alpha} = \approx 9.09\alpha$, that gives almost twice longer time.

2.2. Parallel system

For a system with parallel connected components, each of which has an exponential lifetime distribution with parameters λ_i, $(i = \overline{1,m})$, the reliability function of the system equals (Gertsbakh 2000)

$$R(t) = 1 - \prod_{1\le i\le m}(1 - e^{-\lambda_i t}).$$

Thus, for any given reliability level of the system $r = 1$-α in order to reach the guaranteed lifetime $t_{1-\alpha}$ of the system one should provide the reliability level of i-th component equal $R_i(t_{1-\alpha}) = e^{-\lambda_i t_{1-\alpha}}$. For enough reliable systems with reliability level of components close to one, this gives $r_i = 1 - \alpha_i = e^{-\lambda_i t_{1-\alpha}} \approx 1 - \lambda_i t_{1-\alpha}$, or $\alpha_i \approx \lambda_i t_{1-\alpha}$. This shows that the level of i-th component to fail should be proportional to the failure intensity. One could find the proportionally coefficient c from the equality $\alpha = \prod_{1\le i\le m}\alpha_i = c^m \prod_{1\le i\le m}\lambda_i$. From this equality it follows that

$$c = \left(\frac{\alpha}{\prod_{1\le i\le m}\lambda_i}\right)^{\frac{1}{m}},$$

and thus

$$\alpha_i = \lambda_i\left(\frac{\alpha}{\prod_{1\le i\le m}\lambda_i}\right)^{\frac{1}{m}}.$$

This shows the difference between reliability levels of the components.

These examples show that the reliability level for different components of the system should be different in order to provide maximal guaranteed lifetime of the system. Thus the problem arise how to share of given level of the reliability of a system between its components.

In mathematical terms the problem could be formulated as follows. Suppose that the entity consists of m components with reliability functions $R_i(t)$ $(i = \overline{1,m})$, and has a structure function $f(x)=f(x_1,x_2,\dots,x_n)$. This means that the reliability function of the entity is (see, Gertsbakh 2000).

$$R(t) = \mathbf{E}[f(x_1,x_2,\&,x_m)] = f(R_1(t),R_2(t),\&R_m(t)) \quad (1)$$

Thus, one should choose a point $\mathbf{r} = (r_1, r_2\&, r_m)$ in the hyper-space

$$f(r_1,r_2,\dots,r_m)\ge r = 1\text{-}\alpha \quad \text{with}$$
$$d\{(r_1, r_2\&, r_m) : 0 \le r_i \le 1\ (i = \overline{1,m})\} \quad (2)$$

in such a way to maximize

$$t_{1-\alpha} = R^{-1}(1-\alpha) \Rightarrow \max \quad (3)$$

3. Problems solution

A theoretical solution of the problem is very simple. If one know the reliability function of the system (1) he/she can solve (at least in principle) an equation

$$R(t) = r = 1\text{-}\alpha \quad (4)$$

to find $t_{1-\alpha} = R^{-1}(1$-$\alpha)$. Due to usual strong monotonicity of the function $R(t)$ the solution exists and unique. Thus, the reliability level of each component equals $r_i = 1$-$\alpha_i = R_i(t_{1-\alpha})$.

Nevertheless, because the reliability function $R(t)$ in real world problems is enough complicated and moreover it is composition of several functions: structure function of a system and reliability functions of its components – the exact solution of this equation is really impossible.

Because any monotone system can be represented as a system of consequence-parallel structure we will consider here these types of structures. We propose euristical algorithms for the problem solution for two cases: consequence and parallel systems.

To reliability share for consequence system it is possible to use the following algorithm

3.1. Algorithm 1. Series system

Input initial data:
Integer: m – number of subsystems;
Real: ε – accuracy coefficient, r – consumer's reliability level;
Functions: $R_i(t)$ – reliability functions.
Begin. Find an initial point $\mathbf{r}^{(0)} = (r_1^{(0)},\&, r_m^{(0)})$ at the hyper-space

$$f(\mathbf{r}) = \prod_{1\le i\le m} r_i = r, \quad \{(r_1, r_2\&, r_m) : 0 \le r_i \le 1\ (i = \overline{1,m})\} \quad (5)$$

For series system as an initial point it is possible to take $r_i^{(0)} = r^{1/m}$. Go to the step 1 with $k = 0$.
Step 1. For inverse functions $R_i^{(-1)}(\cdot)$ calculate $t_i^{(k)} = R_i^{(-1)}(r_i^{(k)})$ and arrange them in order to increasing

$$t_{i_1}^{(k)} \le t_{i_2}^{(k)} \le \& \le t_{i_m}^{(k)},$$

where i_j denotes the number of component having j-th in order lifetime.
Step 2. Check if $t_{i_m}^{(k)} - t_{i_1}^{(k)} \le \varepsilon$ go to the step 4, in other case go to the step 3.
Step 3. Change the point $\mathbf{r}$ at the hyper-space (5) in order to decrease r_{i_1} and increase r_{i_m}. For example, with some improvement coefficient $\gamma < 1$ put $r_{i_1}^{k-1} = \gamma r_{i_1}^k$ and $r_{i_m}^{k+1} = \gamma^{-1} r_{i_m}^k$. Change k to k+1. Go to the step 1 with new value of r_i^{k+1}.

Step 4.Print results.
End

For the systems with parallel connection one should work with fail probabilities instead of subsystems reliability. Thus the algorithm looks like this one.

3.2. Algorithm 2. Parallel system

Input initial data:
Integer: m – number of subsystems;
Real: ε – accuracy coefficient, α – probability level for the entity to fail;
Functions: $F_i(t)$ – lifetimes c.d.f.
Begin. Find an initial point $(\alpha_1^{(0)},..., \alpha_m^{(0)})$ with $\alpha_i^{(0)} = 1 - r_i^{(0)}$ at the hyper-space

$$1 - f(\mathrm{r}) = \prod_{1 \le i \le m} \alpha_i = \alpha,$$

$$\{(\alpha_1, \&, \alpha_m) : 0 \le \alpha_i \le 1\ (i = \overline{1,m})\}. \qquad (6)$$

For parallel system as an initial point it is possible to take $\alpha_i^{(0)} = \alpha^{1/m}$. Go to the step 1 with $k = 0$.
Step 1. For inverse functions $F_i^{(-1)}(\cdot)$ calculate $t_i^{(k)} = F_i^{(-1)}(\alpha_i^{(k)})$ and arrange them in order to increasing

$$t_{i_1}^{(k)} \le t_{i_2}^{(k)} \le \& \le t_{i_m}^{(k)}$$

where i_j denotes the number of component having j-th in order lifetime.
Step 2. Check if $t_{i_m}^{(k)} - t_{i_1}^{(k)} \le \varepsilon$ go to the step 4, in other case go to the step 3.
Step 3. Change the point $(\alpha_1^{(k)},..., \alpha_m^{(k)})$ at the hyper-space $\alpha = \prod_{1 \le i \le m} \alpha_i$ in order to increase α_{i_1} and decrease α_{i_m}. For example, with some improvement coefficient $\gamma > 1$ put $\alpha_{i_1}^{k+1} = \gamma \alpha_{i_1}^{k}$ and $\alpha_{i_m}^{k+1} = \gamma^{-1} \alpha_{i_m}^{k}$. Change k to $k+1$. Put $1 - \alpha_i^k = r_i^{k-1}$. Go to the step 1 with new values of α_i^{k+1}.
Step 4. Print results.
End

The results of the algorithms could be formulated as follows: *to increase the lifetime of a system with sequential connection of subsystem one should strengthen the weakest component, while for the system with parallel connection one should strengthen the strongest one.*

In real world problems the exact reliability functions are usually not known. A problem arise on how to use observed data instead of exact information about reliability functions. We propose a statistical approach for solving the above problem.

4. Statistical approach

In practice producers really do not have complete information about the true reliability functions of the components in use. In reality, only some statistical observations about the component's lifetimes are available. Thus, we also propose an approach to the solution of the problem when some statistical or mixed data are available.

Let $t_{i,1}, t_{i,2}, \&, t_{i,n_i}$ $(i = \overline{1,m})$ be the observations on the component's lifetimes ordered in increasing their values separately for each of the components. It is well known that the best estimation for the α-percentile of a distribution is the empirical (sample) percentile, given by the formula $t_{\alpha_i} = t_{i,[\alpha_i n_i]+1}$. Thus, in the above proposed procedure one could use empirical percentiles instead of the theoretical ones when the true lifetime distributions are not available. For this case only the problem arise with the stopping procedure.

Also both cases with consequence and parallel connection should be considered separately. We propose an Algorithm only for consequence connection of a system.

4.1. Algorithm 3. Statistical

Input initial data:
Integer: m — number of subsystems,
Real: r — consumer's reliability level;
Observations: $t_{i,1}, t_{i,2}, \&, t_{i,n_i}$ $(i = \overline{1,m})$ – lifetime of components observations.
Begin. Arrange the observed data in order of increasing values for any component

$$t_{i,1} \le t_{i,2} \le \& \le t_{i,l_i} \le \& \le t_{i,n_i} \quad (i = \overline{1,m})$$

Put $t_i^{(0)} = t_{i,1}$, $l_i^{(0)} = 0$ $(i = \overline{1,m})$, $r_i^{(0)} = 1$. Go to step 1 with $k = 0$.
Step 1. Find $t^{(k)} = \min_{1 \le i \le m} t_i^{(k)}$, $i^{(k)} = \operatorname{argmin}_{1 \le i \le m} t_i^{(k)}$.
Step 2. Check if $l_i^{(k)} \le n_i$ and $r^{(k)} \ge r$ go to step 3 otherwise go to step 4.
Step 3.Change k to $k+1$. Put $l_i^{(k-1)} = l_i^{(k)} + 1$ for $i = i^k$, $t_i^{(k+1)} = t_{i,l_i^{(k+1)}}$ for $i = i^k$. Calculate

$$r^{(k+1)} = \prod_{1 \le i \le m} \frac{n_i - l_i^{(k)}}{n_i} = r^{(k)} \left(1 - \frac{1}{n_i - l_i^{(k)}}\right)$$

Go to the step 1.
Step 4. Print results.
End

5. Conclusion

The proposed approach considers an optimization aspect in reliability systems. It could be realized as a special Computer oriented Project and realized in different branches of industry.

Acknowledgements

One of the authors thanks the Russian Found of Fundamental Investigations for the support of the study due to the Grant Number 04-07-901156.

References

[1] Lisnianski, A. and G.Levitin (2003): *Multi-State System Reliability*. World Scientific.

[2] Dimitrov, B., D.Green, V.Rykov and P.Stanchev (2004): *Reliability of Complex Hierarchical Fault Tolerance Systems*. In Abstracts of The 4-th International Conference on Mathematical Methods in Reliability (MMR-2004), Santa Fe, (USA), 21-25 July, 2004.

[3] Dimitrov, B., D.Green, V.Rykov and P.Stanchev (2004): *Reliability Model for Biological Objects*. In Proceedings of The 1-st International Conference on Longevity, Aging and Degradation in Reliability, Medicine and Biology (LAD-2004), Saint Petersburg, (Russia), 7-9 July, 2004

[4] Dimitrov, B., V. Rykov(2002): *On multi-state reliability systems*. In Applied Stochastic Models and Information Processes. Proceedings of International Seminar. Karelian Research Center of the Russian Academy of Science. Institute of Applied Mathematical Research. Petrozavodsk, 2002. P.128-135.

[5] Dimitrov, B., V.Rykov and P.Stanchev (2002): *On multi-state reliability systems*. In Proceedings of the MMR 2002, Trondheim, Norway 2002. P.136-140

[6] Rykov, V., D. Efrosinin (2004): *Reliability Control of Fault Tolerance Units*. In: Abstracts of The 4-th International Conference on Mathematical Methods in Reliability (MMR-2004), Santa Fe, (USA), 21-25 July, 2004

[7] Ermolaev, A.I. and V.V. Rykov (2000): *The problem of optimal reservation with different types of equipment*. In Second International Conference on Mathematical Methods in Reliability (MMR'2000). University Victor Segalen Bordeaux 2, Vol. 1, pp. 381-385.

[8] Gertsbakh, I. (2000): *Reliability Theory with Application to Preventive Maintenance*. Berlin:Springer.

Ph.D. Prof. Boyan DIMITROV
Ph.D. Prof. David GREEN
Ph.D. Prof. Peter STANCHEV
Kettering University Flint, Michigan, USA
e-mail: bdimitro@kettering.edu
e-mail: dgreen@kettering.edu
e-mail: pstanche@kettering.edu

Ph.D Prof. Vladimir RYKOV
Russian State Oil and Gas University, Moscow, Russia
e-mail: vladimir_rykov@mail.ru