USE SEPARATION OF VARIABLES TO SOLVE ALL THESE PROBLEMS.

1. **BIOLOGY:** When a species grows it eventually reaches its \textit{carrying capacity}, at which time the limited resources are insufficient to support additional growth. Such phenomenon can often be modelled by the IVP
\[
\frac{dP}{dt} = P - P^2, \quad P(0) = P_0
\]  \hspace{1cm} (1)
where \( P = P(t) \) is the population (measured in millions) at time \( t \) (measured in years), and \( P_0 \) is the initial population of the species.

(a) Solve the IVP (1) to obtain population \( P \) explicitly in terms of \( t \) and \( P_0 \). Note: It is known that \( 0 < P < 1 \).

Hint: After separating the ODE, you will have to use partial fraction decomposition to integrate.

(b) Plot (using MAPLE?) the population \( P(t) \) on time interval \( 0 \leq t \leq 15 \) years if the initial population is 1,000 (i.e., \( P_0 = 0.001 \)).

You can see why the graph, called a \textit{logistic curve}, is also called an “S–shaped curve”.

\textbf{Note:} This IVP (1), called the \textit{logistic model}, also models many chemical reaction processes, some atmospheric processes, and many other physical processes.

2. **ANALYTICAL GEOMETRY:** A \textit{geodesic} is the shortest path between two points on a surface. It is well known that the shortest path between two points in a plane is a straight line. Also, cartographers have long known that the shortest path between two points on the earth (a sphere) is a “great circle” — a circle centered at the earth’s center. However, both of these known facts require advanced mathematics, called \textit{calculus of variations}, to prove.

Here we will determine a geodesic on a right circular cone. Consider two points on a right circular cone. By calculus of variations, the geodesic on the cone must satisfy the ODE
\[
a \frac{d\rho}{d\theta} = k \rho \sqrt{\rho^2 - a^2}, \hspace{1cm} (2)
\]
where \( k \) and \( a \) are constants. Here, \( \rho \) is a function of variable \( \theta \).

- Solve the ODE (2) to obtain the solution, and rewrite the solution in the form
\[
\rho = a \sec(k \theta + B),
\]
where \( B \) is an arbitrary constant. This curve is the geodesic (shortest path) between two points on the cone.

\textbf{Note:} Although it is posed in the context of analytical geometry, this problem is similar to those solved by cartographers who study geodesics on the earth and paths of satellites, and also by airlines which determine shortest flight paths from one city to another, taking into account fuel consumption and passenger safety. These problems fall under the branch of mathematics called \textit{optimization}. Optimization problems are of great interest to industry.