Use Laplace transforms to solve the IVP

$$y'' - 5\,y' + 6\,y \;=\; U(t-1)\,, \tag{1}$$

$$y(0) \;=\; 0\,, \qquad y'(0) \;=\; 1\,. \tag{2}$$

1. **Take $\mathcal{L}$ of ODE (1):**

$$\mathcal{L}\{y''\} \;-\; 5\,\mathcal{L}\{y'\} \;+\; 6\,\mathcal{L}\{y\} \;=\; \mathcal{L}\Big\{U(t-1)\Big\}\,, \tag{3}$$

where

$$\begin{aligned}
\mathcal{L}\{y\} \;&=\; Y\,, \\
\mathcal{L}\{y'\} \;&=\; s\,Y - y(0) \;=\; s\,Y - 0 \;=\; s\,Y\,, \\
\mathcal{L}\{y''\} \;&=\; s^2\,Y - s\,y(0) - y'(0) \;=\; s^2\,Y - s\cdot 0 - 1 \;=\; s^2\,Y - 1\,, \\
\mathcal{L}\Big\{U(t-1)\Big\} \;&=\; \frac{1}{s}\,e^{-s} \qquad \text{by the 2TT}\,.
\end{aligned}$$

Substitute these into Eq. (3) to obtain

$$s^2\,Y \;-\; 1 \;-\; 5\,s\,Y \;+\; 6\,Y \;=\; \frac{1}{s}\,e^{-s}\,.$$

2. **Solve for Y:**

$$\begin{aligned}
(s^2 - 5\,s + 6)\,Y \;&=\; 1 + \frac{1}{s}\,e^{-s}\,, \\
(s-2)(s-3)\,Y \;&=\; 1 + \frac{1}{s}\,e^{-s}\,.
\end{aligned}$$

So

$$Y \;=\; \frac{1}{(s-2)(s-3)} \;+\; \frac{1}{s(s-2)(s-3)}\cdot e^{-s}\,.$$

3. **Apply $\mathcal{L}^{-1}$:**

$$\begin{aligned}
y \;&=\; \mathcal{L}^{-1}\{Y\} && (4) \\
&=\; \mathcal{L}^{-1}\left\{ \frac{1}{(s-2)(s-3)} \;+\; \frac{1}{s(s-2)(s-3)}\,e^{-s} \right\} \\
&=\; \mathcal{L}^{-1}\left\{ \frac{1}{(s-2)(s-3)} \right\} \;+\; \mathcal{L}^{-1}\left\{ e^{-s}\,\frac{1}{s(s-2)(s-3)} \right\} \qquad \text{use I2TT on the 2nd term} \\
&=\; \mathcal{L}^{-1}\left\{ \frac{1}{(s-2)(s-3)} \right\} \;+\; U(t-1)\,\mathcal{L}^{-1}\left\{ \frac{1}{s(s-2)(s-3)} \right\}\Bigg|_{t\to t-1} \qquad \text{by the I2TT} && (5)
\end{aligned}$$

NOTE: By partial fraction decomposition,

$$\frac{1}{(s-2)(s-3)} \;=\; \frac{a}{s-2} + \frac{b}{s-3} \;=\; \cdots \text{ (you show) } \cdots \;=\; -\frac{1}{s-2} + \frac{1}{s-3}\,,$$

and

$$\begin{aligned}\frac{1}{s(s-2)(s-3)} &= \frac{a}{s} + \frac{b}{s-2} + \frac{c}{s-3} \\ &= \cdots \text{ (you show) } \cdots \\ &= \frac{1}{6}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{1}{s-2}\right) + \frac{1}{3}\left(\frac{1}{s-3}\right).\end{aligned}$$

So

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s-3)}\right\} &= -\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} \\ &= -e^{2t} + e^{3t}\,,\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s(s-2)(s-3)}\right\} &= \frac{1}{6}\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{3}\,\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} \\ &= \frac{1}{6} - \frac{1}{2}\,e^{2t} + \frac{1}{3}\,e^{3t}\,.\end{aligned}$$

Therefore, by Eq. (5), the solution of the IVP is

$$\begin{aligned}y(t) &= -e^{2t} + e^{3t} + U(t-1)\left[\frac{1}{6} - \frac{1}{2}\,e^{2t} + \frac{1}{3}\,e^{3t}\right]\Bigg|_{t\to t-1} \\ &= -e^{2t} + e^{3t} + U(t-1)\left[\frac{1}{6} - \frac{1}{2}\,e^{2(t-1)} + \frac{1}{3}\,e^{3(t-1)}\right] \\ &= -e^{2t} + e^{3t} + \frac{1}{6}\cdot U(t-1)\left[1 - 3\,e^{2(t-1)} + 2\,e^{3(t-1)}\right].\end{aligned}$$