Use Laplace transforms to solve the IVP

$$y' + y \;=\; f(t)\,, \tag{1}$$

$$y(0) \;=\; 0\,, \tag{2}$$

where

$$f(t) \;=\; \begin{cases} 1\,, & 0 \le t < 1 \\ -1\,, & t \ge 1 \end{cases} \;.$$

First, express the input function $f(t)$ in terms of unit step functions:

$$\begin{aligned} f(t) \;&=\; 1 - 1\,U(t-1) + (-1)\,U(t-1) \\ &=\; 1 - 2\,U(t-1)\,. \end{aligned}$$

1. **Take $\mathcal{L}$ of ODE (1):**

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} \;=\; \mathcal{L}\{f(t)\}\,, \tag{3}$$

where

$$\begin{aligned} \mathcal{L}\{y\} \;&=\; Y\,, \\ \mathcal{L}\{y'\} \;&=\; s\,Y - y(0) \;=\; s\,Y - 0 \;=\; s\,Y\,, \\ \mathcal{L}\{f(t)\} \;&=\; \mathcal{L}\Big\{1 - 2\,U(t-1)\Big\} \;=\; \frac{1}{s} - \frac{2}{s}\,e^{-s}\,. \end{aligned}$$

Substitute these into Eq. (3) to obtain

$$s\,Y + Y \;=\; \frac{1}{s} - \frac{2}{s}\,e^{-s}\,.$$

2. **Solve for Y:**

$$(s+1)\,Y \;=\; \frac{1}{s} - \frac{2}{s}\,e^{-s}\,,$$

so

$$Y \;=\; \frac{1}{s(s+1)} - \frac{2}{s(s+1)}\,e^{-s}\,.$$

3. **Apply $\mathcal{L}^{-1}$:**

$$\begin{aligned} y \;&=\; \mathcal{L}^{-1}\{Y\} \\ &=\; \mathcal{L}^{-1}\left\{ \frac{1}{s(s+1)} - \frac{2}{s(s+1)}\,e^{-s} \right\} \\ &=\; \mathcal{L}^{-1}\left\{ \frac{1}{s(s+1)} \right\} - 2\,\mathcal{L}^{-1}\left\{ e^{-s}\,\frac{1}{s(s+1)} \right\} \qquad \text{use the I2TT on the 2nd term} \\ &=\; \mathcal{L}^{-1}\left\{ \frac{1}{s(s+1)} \right\} - 2\,U(t-1)\,\mathcal{L}^{-1}\left\{ \frac{1}{s(s+1)} \right\}\Bigg|_{t \to t-1} \qquad \text{by the I2TT} \end{aligned}$$

NOTE: By partial fraction decomposition,

$$\frac{1}{s(s+1)} = \frac{a}{s} + \frac{b}{s+1} = \cdots \text{ (you show) } \cdots = \frac{1}{s} - \frac{1}{s+1},$$

so

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{ \frac{1}{s} - \frac{1}{s+1} \right\} - 2\,U(t-1)\,\mathcal{L}^{-1}\left\{ \frac{1}{s} - \frac{1}{s+1} \right\}\Bigg|_{t\to t-1} \\ &= 1 - e^{-t} - 2\,U(t-1)\left[1 - e^{-t} \right]\Bigg|_{t\to t-1} \\ &= 1 - e^{-t} - 2\,U(t-1)\left[1 - e^{-(t-1)} \right] \end{aligned}$$