A 2 kg mass is attached to a spring with spring constant of 338 N/m. The surrounding medium offers a resistance numerically equal to 20 times the velocity. Initially the spring is 4 m above the equilibrium position and given a downward velocity of 56 m/s. Obtain the equation of motion.

Given:

$m = 2$ kg $\quad k = 338$ N/m $\quad \beta = 20$

$x(0) = -4$ m $\quad x'(0) = +56$ m/s

The ODE (governing equation) is then

$$m\,x'' + \beta\,x' + k\,x \;=\; 0 \qquad \Longrightarrow \qquad 2\,x'' + 20\,x' + 338\,x \;=\; 0\,. \tag{1}$$

The characteristic equation (after dividing by 2)

$$m^2 + 10\,m + 169 \;=\; 0$$

has roots

$$m_{1,2} \;=\; -5 \pm 12i\,.$$

Since the roots are complex, the general solution of this homogeneous ODE is *underdamped*:

$$x(t) \;=\; e^{-5t}(c_1\,\cos 12t + c_2\,\sin 12t)\,. \tag{2}$$

So

$$x'(t) \;=\; -5\,e^{-5t}(c_1\,\cos 12t + c_2\,\sin 12t) + e^{-5t}(-12c_1\,\sin 12t + 12c_2\,\cos 12t)\,. \tag{3}$$

Initial Conditions: From equations (2) and (3) we obtain

$$\begin{aligned} x(0) &= e^0(c_1\,\cos 0 + c_2\,\sin 0)\,, \\ -4 &= c_1\,, \end{aligned} \tag{4}$$

$$\begin{aligned} x'(0) &= -5\,e^0(c_1\,\cos 0 + c_2\,\sin 0) + e^0(-12c_1\,\sin 0 + 12c_2\,\cos 0)\,, \\ +56 &= -5c_1 + 12c_2\,. \end{aligned} \tag{5}$$

We solve equations (4) and (5) for c_1 and c_2 to obtain $c_1 = -4$ and $c_2 = 3$. So the *equation of motion* is

$$x(t) \;=\; e^{-5t}(-4\cos 12t + 3\sin 12t)\,. \tag{6}$$

NOTE: The *period* is the time it takes to complete one cycle (vibration, oscillation):

$$T \;=\; \frac{2\pi}{12} \;=\; \frac{\pi}{6} \text{ seconds/cycle}\,, \qquad \text{(takes } \pi/6 \text{ seconds to complete each cycle)}$$

The *frequency* is number of cycles completed every second:

$$f \;=\; \frac{1}{T} \;=\; \frac{6}{\pi} \text{ cycles/sec}\,, \qquad \text{(completes } 6/\pi \text{ cycles per second)}$$

Also, we calculate A (note: A is no longer the amplitude):

$$A \;=\; \sqrt{c_1^2 + c_2^2} \;=\; \sqrt{(-4)^2 + (3)^2} \;=\; 5\,\text{m}\,,$$

so the *damped amplitude* is

$$Ae^{-5t} \;=\; 5e^{-5t}\,\text{m}\,.$$