A circuit comprises a 1 henry inductor with a 0.25 farad capacitor. There is no resistor. Initially $q(0) = 0$ and $i(0) = 0$, but there is an impressed voltage of $E(t) = 24 \sin 2t$. Obtain the charge $q(t)$.

Given:

$L = 1$ h	$R = 0\,\Omega$	$C = 0.25$ f
$q(0) = 0$	$q'(0) = i(0) = 0$	

The nonhomogeneous ODE (governing equation) is then

$$L\,q'' + Rq' + \frac{1}{C}\,q \;=\; E(t) \qquad \Longrightarrow \qquad q'' + 4\,q \;=\; 24 \sin 2t\,. \tag{1}$$

STEP 1: COMPLEMENTARY SOLUTION

The characteristic equation

$$m^2 + 4 \;=\; 0$$

has roots

$$m_{1,2} \;=\; \pm 2i\,.$$

So the complementary solution is (see Sec. 5.1.1)

$$q_c(t) \;=\; c_1\,\cos 2t + c_2\,\sin 2t\,, \tag{2}$$

STEP 2: PARTICULAR SOLUTION

Now we seek a particular solution of (1). We'll use **undetermined coefficients:**

Input Function:	Terms
$E \;=\; 24 \sin 2t$	$\sin 2t$
$E' \;=\; 48 \cos 2t$	$\cos 2t$
$E'' \;=\; -96 \sin 2t$	$\sin 2t$
List:	$\cos 2t,\ \sin 2t$
New List:	$t \cos 2t,\ t \sin 2t$

$\mathcal{Q}$: Do any terms in the List already appear in q_c?
$\mathcal{A}$: Yes, so we must modify the List.

So we seek a particular solution of (1) that is a linear combination of terms in the New List:

$$q_p \;=\; at \cos 2t + bt \sin 2t\,. \tag{3}$$

We substitute q_p into (1) and collect like terms to obtain

$$4b\,\cos 2t - 4a\,\sin 2t \;\equiv\; 24 \sin 2t\,.$$

Equate like terms:

$$\begin{aligned} \cos 4t &: \quad 4b \;\equiv\; 0 \quad \Longrightarrow b = 0 \\ \sin 4t &: \quad -4a \;\equiv\; 24 \quad \Longrightarrow a = -6 \end{aligned}$$

So by (3), a particular solution of (1) is

$$q_p(t) \;=\; -6t\cos 2t\,. \tag{4}$$

STEP 3: GENERAL SOLUTION

Then the general solution of the nonhomogeneous problem (1) is

$$q(t) \;=\; q_c + q_p \;=\; c_1\,\cos 2t + c_2\,\sin 2t - 6t\cos 2t\,. \tag{5}$$

and

$$q'(t) \;=\; -2c_1\,\sin 2t + 2c_2\,\cos 2t - 6\cos 2t + 12t\sin 2t \tag{6}$$

STEP 4: APPLY INITIAL CONDITIONS:

From equations (5) and (6) we obtain

$$q(0) \;=\; c_1 \;\equiv\; 0\,, \tag{7}$$

$$q'(0) \;=\; 2\,c_2 - 6 \;\equiv\; 0 \quad\Longrightarrow\; c_2 \;=\; 3\,. \tag{8}$$

So by (5), the solution is

$$\text{charge:}\qquad q(t) \;=\; 3\,\sin 2t - 6t\cos 2t\,, \tag{9}$$

$$\text{current:}\qquad i(t) \;\equiv\; q'(t) \;=\; 12t\sin 2t\,.$$

NOTES & NEW CONCEPTS:

1. In this example, the complementary solution $q_c(t) \;=\; c_1\,\cos 2t + c_2\,\sin 2t$ represents *simple harmonic motion*, so the *natural frequency* is

$$f \;=\; \frac{2}{2\pi} \;=\; \frac{1}{\pi}\ \text{Hz}\,.$$

2. However, the particular solution $q_p(t) \;=\; -6t\,\cos 2t$ represents *unbounded oscillation*. Notice that its frequency is also

$$f \;=\; \frac{2}{2\pi} \;=\; \frac{1}{\pi}\ \text{Hz}\,.$$

   Note also that the frequency of the particular solution IS the frequency of the driving potential $E(t)$.

3. **RECALL:** When the natural frequency and the frequency of the driving potential $E(t)$ are equal (and no damping is present), then the solution $q(t)$ will grow without bound. This occurred *despite* the fact that the driving potential $E(t)$ is bounded. We call this phenomenon *pure resonance*.

4. This problem is exactly Example 7 that was handed out in Section 5.1.3.