A circuit comprises a 1 henry inductor and a 0.25 farad capacitor. There is no resistor. Initially $q(0) = 0$ and $i(0) = 0$, but there is an impressed voltage of $E(t) = 24 \sin 4t$. Obtain the charge $q(t)$.

Given:

$$L = 1 \text{ h}, \qquad R = 0\,\Omega, \qquad C = 0.25 \text{ f},$$
$$q(0) = 0, \qquad q'(0) = i(0) = 0.$$

The nonhomogeneous ODE (governing equation) is then

$$L\,q'' + Rq' + \frac{1}{C}\,q \;=\; E(t) \qquad \Longrightarrow \qquad q'' + 4\,q \;=\; 24 \sin 4t\,. \tag{1}$$

STEP 1: COMPLEMENTARY SOLUTION

The characteristic equation

$$m^2 + 4 \;=\; 0$$

has roots

$$m_{1,2} \;=\; \pm 2i\,.$$

So the complementary solution is (see Sec. 5.1.1)

$$q_c(t) \;=\; c_1 \cos 2t + c_2 \sin 2t\,. \tag{2}$$

STEP 2: PARTICULAR SOLUTION

Now we seek a particular solution of (1). We'll use **undetermined coefficients:**

Input Function:	Terms
$E \;=\; 24 \sin 4t$	$\sin 4t$
$E' \;=\; 96 \cos 4t$	$\cos 4t$
$E'' \;=\; -284 \sin 4t$	$\sin 4t$
List:	$\cos 4t,\ \sin 4t$

$\mathcal{Q}$: Do any terms in the List already appear in x_c?
$\mathcal{A}$: No, so we need not modify the List.

So we seek a particular solution of (1) that is a linear combination of terms in the List:

$$q_p \;=\; a \cos 4t + b \sin 4t\,. \tag{3}$$

We substitute q_p into (1) and collect like terms to obtain

$$-12a\,\cos 4t + -12b\,\sin 4t \;\equiv\; 24 \sin 4t\,.$$

Equate like terms:

$$\begin{aligned} \cos 4t &: \quad -12a \;\equiv\; 0 \quad \Longrightarrow\; a = 0 \\ \sin 4t &: \quad -12b \;\equiv\; 24 \quad \Longrightarrow\; b = -2 \end{aligned}$$

So by (3), a particular solution of (1) is

$$q_p(t) \;=\; -2\sin 4t\,. \tag{4}$$

STEP 3: GENERAL SOLUTION

Then the general solution of the nonhomogeneous problem (1) is

$$\begin{aligned} q(t) &= q_c + q_p \\ &= c_1\,\cos 2t + c_2\,\sin 2t - 2\sin 4t\,. \end{aligned} \tag{5}$$

and

$$q'(t) \;=\; -2c_1\,\sin 2t + 2c_2\,\cos 2t - 8\cos 4t \tag{6}$$

STEP 4: APPLY INITIAL CONDITIONS:

From equations (5) and (6) we obtain

$$q(0) \;=\; c_1 \;\equiv\; 0\,, \tag{7}$$

$$q'(0) \;=\; 2\,c_2 - 8 \;\equiv\; 0 \quad\Longrightarrow\; c_2 \;=\; 4\,. \tag{8}$$

So by (5), the solution is

$$\begin{aligned} \text{charge:}\quad & q(t) \;=\; 4\,\sin 2t - 2\sin 4t\,, \\ \text{current:}\quad & i(t) \;\equiv\; q'(t) \;=\; 8\,\cos 2t - 8\cos 4t\,. \end{aligned} \tag{9}$$

NOTES & NEW CONCEPTS:

1. In this example, the complementary solution $q_c(t) = c_1\cos 2t + c_2\sin 2t$ represents *simple harmonic motion*, (*i.e.*, constant amplitude oscillation) with *natural frequency*

$$f \;=\; \frac{2}{2\pi} \;=\; \frac{1}{\pi}\ \text{Hz}\,.$$

2. In this example, the particular solution $q_p(t) = -2\sin 4t$ also represents *simple harmonic motion*, (*i.e.*, constant amplitude oscillation) with frequency

$$f \;=\; \frac{4}{2\pi} \;=\; \frac{2}{\pi}\ \text{Hz}\,.$$

   Note also that the frequency of the particular solution IS the frequency of the voltage $E(t)$.

3. This problem is exactly Example 6 that was handed out in Section 5.1.3.