A 2 kg mass is attached to a spring with spring constant of 32 N/m. The surrounding medium offers a resistance numerically equal to 20 times the velocity. Initially the spring is stretched 9 m and given an upward velocity of 6 m/s. Obtain the equation of motion.

Given:

$m = 2$ kg $\qquad k = 32$ N/m $\qquad \beta = 20$

$x(0) = +9$ m $\qquad x'(0) = -6$ m/s

The ODE (governing equation) is then

$$m\,x'' + \beta\,x' + k\,x \;=\; 0 \qquad \Longrightarrow \qquad 2\,x'' + 20\,x' + 32\,x \;=\; 0\,. \tag{1}$$

The characteristic equation

$$m^2 + 10m + 16 \;=\; 0$$

has roots

$$m_1 \;=\; -2\,, \qquad\qquad m_2 \;=\; -8\,.$$

Since the roots are real and distinct, the general solution of this homogeneous ODE is *overdamped*:

$$x(t) \;=\; c_1\,e^{-2t} + c_2\,e^{-8t}\,. \tag{2}$$

Initial Conditions:

First, from (2),

$$\begin{aligned} x(t) &= c_1\,e^{-2t} + c_2\,e^{-8t}\,, \\ x'(t) &= -2\,c_1\,e^{-2t} - 8\,c_2\,e^{-8t}\,, \end{aligned}$$

and so applying the initial conditions, we get

$$\begin{aligned} x(0) &= c_1\,e^{0} + c_2\,e^{0}\,, \\ +9 &= c_1 + c_2\,, \end{aligned} \tag{3}$$

$$\begin{aligned} x'(0) &= -2\,c_1\,e^{0} - 8\,c_2\,e^{0}\,, \\ -6 &= -2\,c_1 - 8\,c_2\,. \end{aligned} \tag{4}$$

We solve equations (3) and (4) for c_1 and c_2 to obtain $\quad c_1 \;=\; 11 \quad$ and $\quad c_2 \;=\; -2$. So the solution (the *equation of motion*) is

$$x(t) \;=\; 11\,e^{-2t} - 2\,e^{-8t}\,. \tag{5}$$