A 1 kg mass is attached to a spring with spring constant of 4 N/m. There is no resistance. Initially the spring is at rest at the equilibrium position. The motion is driven by an external driving force of $F(t) = 24\sin 2t$. Obtain the equation of motion.

Given:

$m = 1$ kg, $k = 4$ N/m, $\beta = 0$,

$x(0) = 0$ m, $x'(0) = 0$ m/s.

The ODE (governing equation) is then

$$m\,x'' + \beta x' + k\,x \;=\; F(t) \qquad \Longrightarrow \qquad x'' + 4\,x \;=\; 24\sin 2t\,. \tag{1}$$

STEP 1: COMPLEMENTARY SOLUTION

The characteristic equation

$$m^2 + 4 \;=\; 0$$

has roots

$$m_{1,2} \;=\; \pm 2i\,.$$

So the complementary solution of Eq. (1) is

$$x_c(t) \;=\; c_1\,\cos 2t + c_2\,\sin 2t\,, \tag{2}$$

that is, x_c is *simple harmonic* with frequency $2/2\pi$ cyc/sec.

STEP 2: PARTICULAR SOLUTION

Now we seek a particular solution of (1). We'll use **undetermined coefficients:**

Input Function:	Terms
$F \;=\; 24\sin 2t$	$\sin 2t$
$F' \;=\; 48\cos 2t$	$\cos 2t$
$F'' \;=\; -96\sin 2t$	$\sin 2t$
List:	$\cos 2t,\ \sin 2t$
New List:	$t\cos 2t,\ t\sin 2t$

$\mathcal{Q}$: Do any terms in the List already appear in x_c?
$\mathcal{A}$: Yes, so we must modify the List.

So we seek a particular solution of (1) that is a linear combination of terms in the New List:

$$x_p \;=\; at\cos 2t + bt\sin 2t\,. \tag{3}$$

We substitute x_p into (1) and collect like terms to obtain

$$4b\,\cos 2t - 4a\,\sin 2t \;\equiv\; 24\sin 2t\,.$$

Equate like terms:

$$\cos 4t \;:\quad 4b \;\equiv\; 0 \quad \Longrightarrow b = 0$$
$$\sin 4t \;:\quad -4a \;\equiv\; 24 \quad \Longrightarrow a = -6$$

So by (3), a particular solution of (1) is

$$x_p(t) \;=\; -6t\cos 2t\,. \tag{4}$$

STEP 3: GENERAL SOLUTION

Then the general solution of the nonhomogeneous problem (1) is

$$x(t) \;=\; x_c + x_p \;=\; c_1\,\cos 2t + c_2\,\sin 2t - 6t\cos 2t\,, \tag{5}$$

and

$$x'(t) \;=\; -2c_1\,\sin 2t + 2c_2\,\cos 2t - 6\cos 2t + 12t\,\sin 2t\,. \tag{6}$$

STEP 4: APPLY INITIAL CONDITIONS:

From equations (5) and (6) we obtain

$$x(0) \;=\; c_1 \;\equiv\; 0\,, \tag{7}$$
$$x'(0) \;=\; 2\,c_2 - 6 \;\equiv\; 0 \quad\Longrightarrow\; c_2 \;=\; 3\,. \tag{8}$$

So by (5), the solution (the *equation of motion*) is

$$x(t) \;=\; 3\,\sin 2t - 6t\,\cos 2t\,. \tag{9}$$

NOTES & NEW CONCEPTS:

1. In this example, the complementary solution

$$x_c(t) \;=\; c_1\,\cos 2t + c_2\,\sin 2t\,,$$

represents *simple harmonic motion*, (*i.e.*, constant amplitude oscillation), so the *natural frequency* is

$$f \;=\; \frac{2}{2\pi} \;=\; \frac{1}{\pi}\ \text{Hz}\,.$$

2. In this example, the particular solution

$$x_p(t) \;=\; -6t\,\cos 2t\,,$$

represents *unbounded oscillation*. Notice that its frequency is also

$$f \;=\; \frac{2}{2\pi} \;=\; \frac{1}{\pi}\ \text{Hz}\,.$$

Note also that the frequency of the particular solution IS the frequency of the driving force $F(t)$.

3. **DEFINITION.** When the natural frequency and the driving force frequency are equal (and no damping is present), then the solution $x(t)$ will grow without bound. This occurred *despite* the fact that the driving force $F(t)$ is bounded.

We call this phenomenon *pure resonance*.