A 1 kg mass is attached to a spring with spring constant of 4 N/m. There is no resistance. Initially the spring is at rest at the equilibrium position. The motion is driven by an external driving force of $F(t) = 24\sin 4t$. Obtain the equation of motion.

Given:

$$m = 1 \text{ kg}, \qquad k = 4 \text{ N/m}, \qquad \beta = 0,$$
$$x(0) = 0 \text{ m}, \qquad x'(0) = 0 \text{ m/s}.$$

The ODE (governing equation) is then

$$m\,x'' + \beta x' + k\,x \;=\; F(t) \qquad \Longrightarrow \qquad x'' + 4\,x \;=\; 24\sin 4t\,. \tag{1}$$

STEP 1: COMPLEMENTARY SOLUTION

The characteristic equation

$$m^2 + 4 \;=\; 0$$

has roots

$$m_{1,2} \;=\; \pm 2i\,.$$

So the complementary solution of Eq. (1) is

$$x_c(t) \;=\; c_1\,\cos 2t + c_2\,\sin 2t\,, \tag{2}$$

that is, x_c is *simple harmonic* with frequency $2/2\pi$ cyc/sec.

STEP 2: PARTICULAR SOLUTION

Now we seek a particular solution of (1). We'll use **undetermined coefficients:**

Input Function:	Terms
$F \;=\; 24\sin 4t$	$\sin 4t$
$F' \;=\; 96\cos 4t$	$\cos 4t$
$F'' \;=\; -284\sin 4t$	$\sin 4t$
List:	$\cos 4t,\ \sin 4t$

$\mathcal{Q}$: Do any terms in the List already appear in x_c?
$\mathcal{A}$: No, so we need not modify the List.

So we seek a particular solution of (1) that is a linear combination of terms in the List:

$$x_p \;=\; a\cos 4t + b\sin 4t\,. \tag{3}$$

We substitute x_p into (1) and collect like terms to obtain

$$-12a\,\cos 4t + -12b\,\sin 4t \;\equiv\; 24\sin 4t\,.$$

Equate like terms:

$$\begin{aligned} \cos 4t &: \quad -12a \;\equiv\; 0 \quad \Longrightarrow\; a = 0 \\ \sin 4t &: \quad -12b \;\equiv\; 24 \quad \Longrightarrow\; b = -2 \end{aligned}$$

So by (3), a particular solution of (1) is

$$x_p(t) \;=\; -2\sin 4t\,. \tag{4}$$

STEP 3: GENERAL SOLUTION

Then the general solution of the nonhomogeneous problem (1) is

$$\begin{aligned} x(t) &= x_c + x_p \\ &= c_1\,\cos 2t + c_2\,\sin 2t - 2\sin 4t\,, \end{aligned} \tag{5}$$

and

$$x'(t) \;=\; -2c_1\,\sin 2t + 2c_2\,\cos 2t - 8\cos 4t\,. \tag{6}$$

STEP 4: APPLY INITIAL CONDITIONS:

From equations (5) and (6) we obtain

$$x(0) \;=\; c_1 \;\equiv\; 0\,, \tag{7}$$

$$x'(0) \;=\; 2\,c_2 - 8 \;\equiv\; 0 \quad\Longrightarrow\; c_2 \;=\; 4\,. \tag{8}$$

So by (5), the solution (the *equation of motion*) is

$$x(t) \;=\; 4\,\sin 2t - 2\sin 4t\,. \tag{9}$$

NOTES & NEW CONCEPTS:

1. In this example, the complementary solution

   $$x_c(t) \;=\; c_1\,\cos 2t + c_2\,\sin 2t\,,$$

   represents *simple harmonic motion*, (*i.e.*, constant amplitude oscillation) with frequency

   $$f \;=\; \frac{2}{2\pi} \;=\; \frac{1}{\pi}\ \text{Hz}\,.$$

2. In this example, the particular solution

   $$x_p(t) \;=\; -2\sin 4t\,,$$

   also represents *simple harmonic motion*, (*i.e.*, constant amplitude oscillation) with frequency

   $$f \;=\; \frac{4}{2\pi} \;=\; \frac{2}{\pi}\ \text{Hz}\,.$$

   Note also that the frequency of the particular solution IS the frequency of the driving force $F(t)$.

3. **DEFINITION.** When the natural frequency and the driving force frequency are different (and no damping is present), we call the frequency of the complementary solution the *natural frequency*.