A 2 kg mass is attached to a spring with spring constant of 272 N/m. The surrounding medium offers a resistance numerically equal to 24 times the velocity. The initially 4 m below the equilibrium position and given a downward velocity of 36 m/s. The motion is driven by an external driving force of $F(t) = 1200 \sin 8t$. Obtain the equation of motion.

Given:

$$m = 2 \text{ kg}, \qquad k = 272 \text{ N/m}, \qquad \beta = 24,$$
$$x(0) = +4 \text{ m}, \qquad x'(0) = +36 \text{ m/s}.$$

The ODE (governing equation) is then

$$m\,x'' + \beta\,x' + k\,x \;=\; F(t) \;\Longrightarrow\; 2\,x'' + 24\,x' + 272\,x \;=\; 1200 \sin 8t\,,$$
$$\Longrightarrow\; x'' + 12\,x' + 136\,x \;=\; 600 \sin 8t\,. \tag{1}$$

STEP 1: COMPLEMENTARY SOLUTION

The characteristic equation

$$m^2 + 12m + 136 \;=\; 0$$

has roots

$$m_{1,2} \;=\; -6 \pm 10i\,.$$

Since the roots are complex, the complementary solution of Eq. (1) is *underdamped*:

$$x_c(t) \;=\; e^{-6t}(c_1 \cos 10t + c_2 \sin 10t)\,. \tag{2}$$

STEP 2: PARTICULAR SOLUTION

Now we seek a particular solution of (1). We'll use **undetermined coefficients:**

Input Function:	Terms
$f \;=\; 600 \sin 8t$	$\sin 8t$
$f' \;=\; 4800 \cos 8t$	$\cos 8t$
$f'' \;=\; 38400 \sin 8t$	$\sin 8t$
List:	$\cos 8t,\ \sin 8t$

$\mathcal{Q}$: Do any terms in the List already appear in x_c?
$\mathcal{A}$: No, so we need not modify the List.

So we seek a particular solution of (1) that is a linear combination of terms in the List:

$$x_p \;=\; a \cos 8t + b \sin 8t\,. \tag{3}$$

We substitute x_p into (1) and collect like terms to obtain

$$(72a + 96b)\,\cos 8t + (-96a + 72b)\,\sin 8t \;\equiv\; 600 \sin 8t\,.$$

Equate like terms:

$$\begin{aligned} \cos 8t &: \quad 72a + 96b \equiv 0 \\ \sin 8t &: \quad -96a + 72b \equiv 600 \end{aligned}$$

We solve these to obtain $a = -4$ and $b = 3$.

So by (3), a particular solution of (1) is

$$x_p(t) = -4\cos 8t + 3\sin 8t\,. \tag{4}$$

STEP 3: GENERAL SOLUTION

Then the general solution of the nonhomogeneous problem (1) is

$$\begin{aligned} x(t) &= x_c + x_p \\ &= e^{-6t}(c_1\cos 10t + c_2\sin 10t)\ - 4\cos 8t + 3\sin 8t\,, \end{aligned} \tag{5}$$

and

$$\begin{aligned} x'(t) \;=\; & -6e^{-6t}(c_1\cos 10t + c_2\sin 10t) + e^{-6t}(-10\,c_1\sin 10t + 10\,c_2\cos 10t) \\ & + 32\sin 8t + 24\cos 8t\,. \end{aligned} \tag{6}$$

STEP 4: APPLY INITIAL CONDITIONS:

The initial conditions were

$$x(0) = 4 \qquad \text{and} \qquad x'(0) = 36\,,$$

so from equations (5) and (6) we obtain

$$x(0) = c_1 - 4 \equiv 4\,, \tag{7}$$

$$x'(0) = -6\,c_1 + 10\,c_2 + 24 \equiv 36\,. \tag{8}$$

We solve equations (7) and (8) for c_1 and c_2 to obtain $c_1 = 8$ and $c_2 = 6$. So the solution (the *equation of motion*) is

$$x(t) = e^{-6t}\Big(8\cos 10t + 6\sin 10t\Big) - 4\cos 8t + 3\sin 8t\,. \tag{9}$$

NOTES & NEW CONCEPTS:

1. In this example, the complementary solution $x_c(t)$ (see Eq. (2)) represents *decaying oscillation*, *i.e.*,

$$x_c(t) \to 0 \quad \text{as } t \to \infty\,.$$

2. In this example, the particular solution $x_p(t)$ (see Eq. (4)) represents *simple harmonic motion*, *i.e.*, constant amplitude oscillation. It does not decay to zero as $t \to \infty$ but oscillates with constant amplitude for all time.

3. When these events occur, we call
   - $x_c(t)$ a *transient term* since it decays to zero, and
   - $x_p(t)$ a *steady state term*.

In other words, in this example if time t is sufficiently large, then we may approximate the equation of motion $x(t)$, given by Eq. (9), by the particular solution x_p:

$$x(t) \;\approx\; x_p(t) \;=\; -4\cos 8t + 3\sin 8t\,.$$

Thus, if we're interested only in the steady state solution (the long term behavior of the system), then we merely need to perform Steps 1 and 2 and *not do Steps 3 and 4 at all — BIG time saver!*

Thus:

- The *steady state amplitude* (of oscillation) is

$$A_{ss} \;=\; A_p \;=\; \sqrt{(-4)^2 + (3)^2} \;=\; \sqrt{25}\,\text{m} \;=\; 5\,\text{m}\,.$$

- The *steady state period* (of oscillation) is

$$T_{ss} \;\equiv\; \frac{2\pi}{8} \;=\; \frac{\pi}{4}\;\text{sec/cycle}\,,$$

  which is the same as the period of the driving force $F(t)$.

- The *steady state frequency* (of oscillation) is

$$f_{ss} \;\equiv\; \frac{1}{T} \;=\; \frac{4}{\pi}\;\text{cycles/sec} \;=\; \frac{4}{\pi}\;\text{Hz}\,,$$

  which is the same as the frequency of the driving force $F(t)$.