A 2 kg mass is attached to a spring with spring constant of 50 N/m. The surrounding medium offers a resistance numerically equal to 20 times the velocity. Initially the spring is stretched 1 m and given a downward velocity of 10 m/s. Obtain the equation of motion.

Given:

$$m = 2 \text{ kg} \qquad k = 50 \text{ N/m} \qquad \beta = 20$$
$$x(0) = +1 \text{ m} \qquad x'(0) = +10 \text{ m/s}$$

The ODE (governing equation) is then

$$m\,x'' + \beta\,x' + k\,x \;=\; 0 \qquad \Longrightarrow \qquad 2\,x'' + 20\,x' + 50\,x \;=\; 0 \tag{1}$$

The characteristic equation

$$m^2 + 10m + 25 \;=\; 0$$

has roots

$$m_1 \;=\; -5\,, \qquad m_2 \;=\; -5\,.$$

Since the roots repeat, the general solution of this homogeneous ODE is solution is *critically damped*:

$$\begin{aligned} x(t) &= c_1\,e^{-5t} + c_2\,t\,e^{-5t} \\ &= e^{-5t}(c_1 + c_2\,t) \end{aligned} \tag{2}$$

Initial Conditions:

First, from (2),

$$\begin{aligned} x(t) &= e^{-5t}(c_1 + c_2\,t)\,, \\ x'(t) &= -5\,e^{-5t}(c_1 + c_2\,t) + c_2\,e^{-5t}\,, \end{aligned}$$

and so applying the initial conditions, we get

$$\begin{aligned} x(0) &= e^0(c_1 + c_2 \cdot 0)\,, \\ +1 &= c_1 \end{aligned} \tag{3}$$

and

$$\begin{aligned} x'(0) &= -5\,e^0(c_1 + c_2 \cdot 0) + c_2\,e^0\,, \\ +10 &= -5\,c_1 + c_2\,. \end{aligned} \tag{4}$$

We solve equations (3) and (4) for c_1 and c_2 to obtain $c_1 = 1$ and $c_2 = 15$. So the solution (the *equation of motion*) is

$$x(t) \;=\; e^{-5t}(1 + 15\,t) \tag{5}$$

Note: $x = 0$ only when $t = -1/15$, so the mass never crosses the equilibrium position.