The method of **Variation of Parameters** is used to obtain the general solution of the ODE

$$y'' + 4\,y \;=\; 4\tan 2x\,. \qquad \textbf{(N)}$$

IMPORTANT: Note that the first term is a lone y'', so $f(x) = 4\tan 2x$.

STEP 1:
Obtain the two linearly independent solutions y_1 and y_2 of the *associated homogeneous problem*

$$y'' + 4\,y \;=\; 0\,. \qquad \textbf{(H)}$$

They are

$$y_1 \;=\; \cos 2x\,, \qquad\qquad y_2 \;=\; \sin 2x\,,$$

so the *complementary solution* of (N) is

$$y_c \;=\; c_1 y_1 + c_2 y_2 \;=\; c_1\cos 2x + c_2\sin 2x\,. \qquad \textbf{(C)}$$

STEP 2: Use **Variation of Parameters** to construct a *particular solution* y_p of problem (N):
First,

$$\begin{aligned} y_1 &= \cos 2x\,, & y_2 &= \sin 2x\,,\\ y_1' &= -2\sin 2x\,, & y_2' &= 2\cos 2x\,. \end{aligned}$$

(a) Set:

$$\begin{aligned} W &\equiv y_1\,y_2' - y_1'\,y_2\\ &= (\cos 2x)(2\cos 2x) - (-2\sin 2x)(\sin 2x) \;=\; 2\\ W_1 &\equiv -y_2\,f \;=\; -(\sin 2x)(4\tan 2x) \;=\; -4\,\frac{\sin^2 2x}{\cos 2x}\\ &= -4\left(\frac{1-\cos^2 2x}{\cos 2x}\right) \;=\; 4\,(\cos 2x - \sec 2x)\,,\\ W_2 &\equiv +y_1\,f \;=\; (\cos 2x)(4\tan 2x) \;=\; 4\sin 2x\,. \end{aligned}$$

(b) Set

$$\begin{aligned} u_1 &\equiv \int \frac{W_1}{W}\,dx \;=\; 2\int(\cos 2x - \sec 2x)\,dx \;=\; \sin 2x - \ln|\sec 2x + \tan 2x|\,,\\ u_2 &\equiv \int \frac{W_2}{W}\,dx \;=\; \int \frac{4\sin 2x}{2}\,dx \;=\; -\cos 2x\,. \end{aligned}$$

(c) Then a *particular solution* of (N) is

$$\begin{aligned} y_p &= u_1\,y_1 + u_2\,y_2\\ &= \Big(\sin 2x - \ln|\sec 2x + \tan 2x|\Big)(\cos 2x) \;+\; (-\cos 2x)(\sin 2x)\,,\\ y_p &= -\cos 2x\;\ln|\sec 2x + \tan 2x| \end{aligned} \qquad \textbf{(P)}$$

STEP 3: Then the *general solution* of the nonhomogeneous problem (N) is

$$y \;=\; y_c + y_p \;=\; c_1\cos 2x + c_2\sin 2x \;-\; \cos 2x\;\ln|\sec 2x + \tan 2x| \qquad \textbf{(G)}$$
