The method of **Variation of Parameters** is used to obtain the general solution of the ODE

$$y'' - 2y' + 2\,y \;=\; e^x \sec x\,. \qquad \textbf{(N)}$$

IMPORTANT: Note that the first term is a lone y'', so $f(x) = e^x \sec x$.

STEP 1:
Obtain the two linearly independent solutions y_1 and y_2 of the *associated homogeneous problem*

$$y'' - 2y' + 2\,y \;=\; 0\,. \qquad \textbf{(H)}$$

They are

$$y_1 \;=\; e^x \cos x\,, \qquad\qquad y_2 \;=\; e^x \sin x\,,$$

so the *complementary solution* of (N) is

$$y_c \;=\; c_1 y_1 + c_2 y_2 \;=\; c_1 e^x \cos x + c_2 e^x \sin x\,. \qquad \textbf{(C)}$$

STEP 2: Use **Variation of Parameters** to construct a *particular solution* y_p of problem (N):
First,

$$y_1 \;=\; e^x \cos x\,, \qquad\qquad y_2 \;=\; e^x \sin x\,,$$

$$y_1' \;=\; e^x \cos x - e^x \sin x\,, \qquad\qquad y_2' \;=\; e^x \sin x + e^x \cos x\,.$$

(a) Set:

$$\begin{aligned}
W &\equiv y_1\, y_2' - y_1'\, y_2 \\
&= (e^x \cos x)\Big(e^x \sin x + e^x \cos x\Big) - \Big(e^x \cos x - e^x \sin x\Big)(e^x \sin x) \\
&= e^{2x}\,, \\
W_1 &\equiv -y_2\, f \;=\; -(e^x \sin x)(e^x \sec x) \;=\; -\,e^{2x} \tan x\,, \\
W_2 &\equiv +y_1\, f \;=\; (e^x \cos x)(e^x \sec x) \;=\; e^{2x}\,.
\end{aligned}$$

(b) Set

$$u_1 \;\equiv\; \int \frac{W_1}{W}\,dx \;=\; \int \frac{-\,e^{2x}\tan x}{e^{2x}}\,dx \;=\; -\int \tan x\,dx \;=\; \ln|\cos x|\,,$$

$$u_2 \;\equiv\; \int \frac{W_2}{W}\,dx \;=\; \int \frac{e^{2x}}{e^{2x}}\,dx \;=\; \int 1\,dx \;=\; x\,.$$

(c) Then a *particular solution* of (N) is

$$y_p \;=\; u_1\, y_1 + u_2\, y_2 \;=\; e^x \cos x\, \ln|\cos x| \;+\; x e^x \sin x \qquad \textbf{(P)}$$

STEP 3: Then the *general solution* of the nonhomogeneous problem (N) is

$$y \;=\; y_c + y_p \;=\; c_1\, e^x \cos x + c_2\, e^x \sin x \;+\; e^x \cos x\, \ln|\cos x| \;+\; x e^x \sin x \qquad \textbf{(G)}$$
