The method of **Variation of Parameters** is used to obtain the general solution of the ODE

$$y'' - y' - 6\,y \;=\; 3\,e^{-x} - 4\,e^{3x}\,. \qquad \textbf{(N)}$$

IMPORTANT: Note that the first term is a lone y'', so $f(x) = 3\,e^{-x} - 4\,e^{3x}$.

STEP 1:
Obtain the two linearly independent solutions y_1 and y_2 of the *associated homogeneous problem*

$$y'' - y' - 6\,y \;=\; 0\,. \qquad \textbf{(H)}$$

We already did this; they are

$$y_1 \;=\; e^{3x}\,, \qquad\qquad y_2 \;=\; e^{-2x}\,,$$

so the *complementary solution* of (N) is

$$y_c \;=\; c_1\,y_1 + c_2\,y_2 \;=\; c_1\,e^{3x} + c_2\,e^{-2x}\,. \qquad \textbf{(C)}$$

STEP 2: Use **Variation of Parameters** to construct a *particular solution* y_p of problem (N):
First,

$$y_1 \;=\; e^{3x}\,, \qquad\qquad y_2 \;=\; e^{-2x}\,,$$
$$y_1' \;=\; 3e^{3x}\,, \qquad\qquad y_2' \;=\; -2e^{-2x}\,.$$

(a) Set:

$$\begin{aligned} W &\equiv y_1\,y_2' - y_1'\,y_2 \\ &= (e^{3x})(-2e^{-2x}) - (3e^{3x})(e^{-2x}) \;=\; -5e^{x}\,, \\ W_1 &\equiv -y_2\,f \;=\; -(e^{-2x})\left(3\,e^{-x} - 4\,e^{3x}\right) \;=\; -3\,e^{-3x} + 4\,e^{x}\,, \\ W_2 &\equiv +y_1\,f \;=\; (e^{3x})\left(3\,e^{-x} - 4\,e^{3x}\right) \;=\; 3\,e^{2x} - 4\,e^{6x}\,. \end{aligned}$$

(b) Set

$$u_1 \;\equiv\; \int \frac{W_1}{W}\,dx \;=\; \int \frac{-3\,e^{-3x} + 4\,e^{x}}{-5e^{x}}\,dx \;=\; \frac{1}{5}\int \left(3\,e^{-4x} - 4\right)dx \;=\; -\frac{3}{20}\,e^{-4x} - \frac{4}{5}\,x\,,$$
$$u_2 \;\equiv\; \int \frac{W_2}{W}\,dx \;=\; \int \frac{3\,e^{2x} - 4\,e^{6x}}{-5e^{x}}\,dx \;=\; -\frac{1}{5}\int \left(3\,e^{x} - 4\,e^{5x}\right)dx \;=\; -\frac{3}{5}\,e^{x} + \frac{4}{25}\,e^{5x}\,.$$

(c) Then a *particular solution* of (N) is

$$y_p \;=\; u_1\,y_1 + u_2\,y_2 \;=\; \left(-\frac{3}{20}\,e^{-4x} - \frac{4}{5}\,x\right)(e^{3x}) \;+\; \left(-\frac{3}{5}\,e^{x} + \frac{4}{25}\,e^{5x}\right)(e^{-2x})$$

$$y_p \;=\; -\frac{3}{4}\,e^{-x} + \frac{4}{25}\,e^{3x} - \frac{4}{5}\,x\,e^{3x}\,. \qquad \textbf{(P)}$$

STEP 3: Then the *general solution* of the nonhomogeneous problem (N) is

$$y \;=\; y_c + y_p \;=\; c_1\,e^{3x} + c_2\,e^{-2x} - \frac{3}{4}\,e^{-x} + \frac{4}{25}\,e^{3x} - \frac{4}{5}\,x\,e^{3x}\,,$$

$$y \;=\; c_1\,e^{3x} + c_2\,e^{-2x} - \frac{3}{4}\,e^{-x} - \frac{4}{5}\,x\,e^{3x}\,. \qquad \textbf{(G)}$$
