The method of **Variation of Parameters** is used to obtain the general solution of the ODE

$$y'' - y' - 6\,y \;=\; e^{x}\,. \qquad \textbf{(N)}$$

IMPORTANT: Note that the first term is a lone y'', so $f(x) = e^x$.

STEP 1:
Obtain the two linearly independent solutions y_1 and y_2 of the *associated homogeneous problem*

$$y'' - y' - 6\,y \;=\; 0\,. \qquad \textbf{(H)}$$

We already did this; they are

$$y_1 \;=\; e^{3x}\,, \qquad\qquad y_2 \;=\; e^{-2x}\,,$$

so the *complementary solution* of (N) is

$$y_c \;=\; c_1\,y_1 + c_2\,y_2 \;=\; c_1\,e^{3x} + c_2\,e^{-2x}\,. \qquad \textbf{(C)}$$

STEP 2: Use **Variation of Parameters** to construct a *particular solution* y_p of problem (N):
First,

$$y_1 \;=\; e^{3x}\,, \qquad\qquad y_2 \;=\; e^{-2x}\,,$$
$$y_1' \;=\; 3e^{3x}\,, \qquad\qquad y_2' \;=\; -2e^{-2x}\,.$$

(a) Set:

$$\begin{aligned} W &\equiv y_1\,y_2' - y_1'\,y_2 \\ &= (e^{3x})(-2e^{-2x}) - (3e^{3x})(e^{-2x}) \;=\; -5e^{x}\,, \\ W_1 &\equiv -y_2\,f \;=\; -(e^{-2x})(e^{x}) \;=\; -e^{-x}\,, \\ W_2 &\equiv +y_1\,f \;=\; (e^{3x})(e^{x}) \;=\; e^{4x}\,. \end{aligned}$$

(b) Set

$$\begin{aligned} u_1 &\equiv \int \frac{W_1}{W}\,dx \;=\; \int \frac{-e^{-x}}{-5e^{x}}\,dx \;=\; \frac{1}{5}\int e^{-2x}\,dx \;=\; -\frac{1}{10}\,e^{-2x}\,, \\ u_2 &\equiv \int \frac{W_2}{W}\,dx \;=\; \int \frac{e^{4x}}{-5e^{x}}\,dx \;=\; -\frac{1}{5}\int e^{3x}\,dx \;=\; -\frac{1}{15}\,e^{3x}\,. \end{aligned}$$

(c) Then a *particular solution* of (N) is

$$y_p \;=\; u_1\,y_1 + u_2\,y_2 \;=\; \Big(-\frac{1}{10}e^{-2x}\Big)(e^{3x}) \;+\; \Big(-\frac{1}{15}e^{3x}\Big)(e^{-2x}) \;=\; -\frac{1}{6}\,e^{x}\,. \qquad \textbf{(P)}$$

STEP 3: Then the *general solution* of the nonhomogeneous problem (N) is

$$y \;=\; y_c + y_p \;=\; c_1\,e^{3x} + c_2\,e^{-2x} - \frac{1}{6}\,e^{x}\,. \qquad \textbf{(G)}$$

STEP 4: Apply any initial conditions to the general solution (G), **NOT** to solution (C).