Solve the nonhomogeneous ODE

$$y'' - 2y' + 5y \;=\; e^x \cos 2x\,, \qquad \textbf{(N)}$$

NOTE: The input function is $g(x) = e^x \cos 2x$.

STEP 1:
Solve the *associated homogeneous problem*:

$$y'' - 2y' + 5y \;=\; 0\,, \qquad \textbf{(H)}$$

to obtain the *complementary solution* of (N):

$$y_c \;=\; c_1\, e^x \cos 2x + c_2\, e^x \sin 2x\,. \qquad \textbf{(C)}$$

STEP 2:
Construct a *particular solution* y_p of (N) by **Undetermined Coefficients**.

Input Function:	Terms
$g \;=\; e^x \cos 2x$	$e^x \cos 2x$
$g' \;=\; e^x \cos 2x - 2\, e^x \sin 2x$	$e^x \cos 2x, \quad e^x \sin 2x$
$g'' \;=\; -3\, e^x \cos 2x - 4\, e^x \sin 2x$	$e^x \cos 2x, \quad e^x \sin 2x$
List:	$e^x \cos 2x, \quad e^x \sin 2x$
New List:	$x\, e^x \cos 2x, \quad x\, e^x \sin 2x$

$\mathcal{Q}$: Do any terms in the List already appear in y_c?
$\mathcal{A}$: Yes, both terms $e^x \cos 2x$ and $e^x \sin 2x$ already appear in y_c, so must modify the List:

$$e^x \cos x \;\rightarrow\; x\, e^x \cos x\,, \qquad e^x \sin x \;\rightarrow\; x\, e^x \sin x\,.$$

So we seek a particular solution of (N) that is a linear combination of terms in the New List:

$$y_p \;=\; ax\, e^x \cos 2x + bx\, e^x \sin 2x\,. \qquad \textbf{(P)}$$

We will substitute y_p into (N), but first

$$y_p \;=\; ax\, e^x \cos 2x + bx\, e^x \sin 2x\,,$$

$$\begin{aligned} y_p' \;&=\; ae^x \cos 2x + ax\, e^x \cos 2x - 2ax\, e^x \sin 2x \\ &\quad + \; be^x \sin 2x + bx\, e^x \sin 2x + 2bx\, e^x \cos 2x\,, \qquad \text{(note symmetry)} \\ &=\; ae^x \cos 2x + be^x \sin 2x + (a+2b)xe^x \cos 2x + (-2a+b)xe^x \sin 2x\,, \end{aligned}$$

$$\begin{aligned} y_p'' \;&=\; 2ae^x \cos 2x - 4ae^x \sin 2x - 3axe^x \cos 2x - 4axe^x \sin 2x \\ &\quad + 2be^x \sin 2x + 4be^x \cos 2x - 3bxe^x \sin 2x + 4bxe^x \cos 2x\,. \qquad \text{(note symmetry)} \\ &=\; (2a+4b)e^x \cos 2x + (-4a+2b)e^x \sin 2x + (-3a+4b)xe^x \cos 2x + (-4a-3b)xe^x \sin 2x\,. \end{aligned}$$

Plug these into (N) to obtain

$$y'' - 2y' + 5y \;=\; e^x \cos 2x \,,$$

$$\begin{aligned}
&(2a+4b)e^x \cos 2x + (-4a+2b)e^x \sin 2x + (-3a+4b)xe^x \cos 2x + (-4a-3b)xe^x \sin 2x \\
&- \;\; 2\left[ae^x \cos 2x + be^x \sin 2x + (a+2b)xe^x \cos 2x + (-2a+b)xe^x \sin 2x\right] \\
&+ \;\; 5\left\{ax\, e^x \cos 2x + bx\, e^x \sin 2x\right\} \qquad \equiv \qquad e^x \cos 2x \,.
\end{aligned}$$

Collect like terms:

$$\begin{aligned}
&(2a+4b-2a)e^x \cos 2x \;+\; (-4a+2b-2b)e^x \sin 2x \\
&\quad + (-3a+4b-2a-4b+5a)xe^x \cos 2x \qquad\qquad \text{(note symmetry)} \\
&\quad + (-4a-3b+4a-2b+5b)xe^x \sin 2x \quad \equiv \quad e^x \cos 2x \,.
\end{aligned}$$

Simplify:

$$4be^x \cos 2x - 4ae^x \sin 2x \quad \equiv \quad e^x \cos 2x \,. \qquad\qquad \text{(note symmetry)}$$

Equate like terms:

$$\begin{aligned}
e^x \cos 2x \;&:\; \quad 4b \;\equiv\; 1 \qquad \Longrightarrow \quad b \;=\; 1/4 \\
e^x \sin 2x \;&:\; \quad -4a \;\equiv\; 0 \quad \Longrightarrow \quad a \;=\; 0
\end{aligned}$$

Litmus Test: Note that these terms are exactly those terms that were in the "List".

So by (P), a particular solution of (N) is

$$\begin{aligned}
y_p \;&=\; axe^x \cos 2x + bxe^x \sin 2x \\
&=\; \frac{1}{4}xe^x \sin 2x \,.
\end{aligned}$$

STEP 3:
Then the general solution of the nonhomogeneous problem (N) is

$$\begin{aligned}
y \;&=\; y_c + y_p \\
&=\; c_1\, e^x \cos 2x + c_2\, e^x \sin 2x + \frac{1}{4}xe^x \sin 2x \,.
\end{aligned}$$

It is a 2–parameter family of solutions of (N).

COMMENTS:

1. Note the numerical "symmetry" at most every step when trig functions are involved.
2. Note that trig functions *always* come in pairs:
   - if there's a $\cos 2x$, there will also be a $\sin 2x$
   - if there's a $e^{3x} \cos 2x$, there will also be a $e^{3x} \sin 2x$
   - if there's a $x^4 \cos 2x$, there will also be a $x^4 \sin 2x$