Solve the nonhomogeneous ODE

$$y'' + 4y = -16\sin 2x\,, \qquad \textbf{(N)}$$

with initial conditions

$$y(0) = -3\,, \qquad y'(0) = 10\,.$$

NOTE: The input function is $g(x) = -16\sin 2x$.

STEP 1:
Solve the *associated homogeneous problem*:

$$y'' + 4y = 0\,, \qquad \textbf{(H)}$$

to obtain the *complementary solution* of (N):

$$y_c = c_1\cos 2x + c_2\sin 2x\,. \qquad \textbf{(C)}$$

STEP 2:
Construct a *particular solution* y_p of (N) by **Undetermined Coefficients**.

Input Function:	Terms
$g = -16\sin 2x$	$\sin 2x$
$g' = -32\cos 2x$	$\cos 2x$
$g'' = 64\sin 2x$	$\sin 2x$
List:	$\cos 2x,\ \sin 2x$
New List:	$x\cos 2x,\ x\sin 2x$

$\mathcal{Q}$: Do any terms in the List already appear in y_c?
$\mathcal{A}$: Yes, so must modify the repeating terms:

$$\cos x \to x\cos x\,, \qquad \sin x \to x\sin x\,.$$

So a particular solution y_p of (N) must be a linear combination of terms in the New List:

$$y_p = ax\cos 2x + bx\sin 2x\,. \qquad \textbf{(P)}$$

We will substitute y_p into (N), but first

$$\begin{aligned} y_p &= ax\cos 2x + bx\sin 2x\,, \\ y_p' &= a\cos 2x - 2ax\sin 2x + b\sin 2x + 2bx\cos 2x\,, \\ y_p'' &= -4a\sin 2x - 4ax\cos 2x + 4b\cos 2x - 4bx\sin 2x\,. \end{aligned}$$

Plug these into (N) to obtain

$$y_p'' + 4y_p \equiv -16\sin 2x\,,$$

$$\begin{aligned} -4a\,\sin 2x - 4ax\,\cos 2x + 4b\,\cos 2x - 4b\,x\sin 2x & \\ +\quad 4\,(\,a\,x\cos 2x + b\,x\sin 2x\,) \quad &\equiv \quad -16\sin 2x\,. \end{aligned}$$

Collect like terms:

$$4b\,\cos 2x - 4a\,\sin 2x + (-4a+4a)\,x\cos 2x + (-4b+4b)\,x\sin 2x \quad \equiv \quad -16\sin 2x\,.$$

simplify:

$$4b\,\cos 2x - 4a\,\sin 2x \quad \equiv \quad 0\,\cos 2x \,+\, -16\,\sin 2x\,.$$

Equate like terms:

$$\begin{aligned} \cos 2x &: \quad 4b \equiv 0 && \Longrightarrow \quad b = 0 \\ \sin 2x &: \quad -4a \equiv -16 && \Longrightarrow \quad a = 4 \end{aligned}$$

Litmus Test: Note that these terms are exactly those terms that were in the "List".

So by (P), a particular solution of (N) is

$$\begin{aligned} y_p &= ax\cos 2x + bx\sin 2x \\ &= 4x\cos 2x\,. \end{aligned}$$

STEP 3:
Then the general solution of the nonhomogeneous problem (N) is

$$\begin{aligned} y &= y_c + y_p \\ &= c_1\cos 2x + c_2\sin 2x + 4x\cos 2x\,. \end{aligned}$$

It is a 2–parameter family of solutions of (N).

STEP 4:
Apply the initial conditions: $\quad y(0) = -3\,, \quad y'(0) = 10\,.$
So,

$$\begin{aligned} y(0) &= c_1\cos 0 + c_2\sin 0 + 4\cdot 0\cdot\cos 0\,, \\ -3 &= c_1\,, \\ \text{and} \quad y'(0) &= -2c_1\sin 0 + 2c_2\cos 0 + 4\cdot\cos 0 - 8\cdot 0\cdot\sin 0\,, \\ 10 &= 2c_2 + 4\,. \end{aligned}$$

So $\quad c_1 = -3 \quad$ and $\quad c_2 = 3$. So the initial value problem (IVP) has solution

$$y = -3\cos 2x + 3\sin 2x + 4x\cos 2x\,.$$