Solve the nonhomogeneous ODE

$$y'' - y' - 6y = 250\,x\cos x\,. \qquad \textbf{(N)}$$

NOTE: The input function is $g(x) = 250\,x\cos x$.

STEP 1:
Solve the *associated homogeneous problem*:

$$y'' - y' - 6y = 0\,. \qquad \textbf{(H)}$$

We did this in Example 1, so the *complementary solution* of (N) is

$$y_c = c_1\,e^{3x} + c_2\,e^{-2x}\,. \qquad \textbf{(C)}$$

STEP 2:
Construct a *particular solution* y_p of (N) by **Undetermined Coefficients**.

Input Function:		Terms
$g = 250\,x\cos x$		$x\cos x$
$g' = 250\cos x - 250\,x\sin x$		$\cos x,\ x\sin x$
$g'' = -500\sin x - 250\,x\cos x$		$\sin x,\ x\cos x$
	List:	$\cos x,\ \sin x,\ x\cos x,\ x\sin x$

$\mathcal{Q}$: Do any terms in the List already appear in y_c?
$\mathcal{A}$: No, so we need not modify any terms in the List.

So a particular solution y_p of (N) must be a linear combination of terms in the List:

$$y_p = a\cos x + b\sin x + c\,x\cos x + d\,x\sin x\,. \qquad \textbf{(P)}$$

We will substitute y_p into (N), but first

$$y_p = a\cos x + b\sin x + c\,x\cos x + d\,x\sin x\,,$$

$$y_p' = -a\sin x + b\cos x + c\cos x - c\,x\sin x + d\sin x + d\,x\cos x\,,$$

$$y_p'' = -a\cos x - b\sin x - 2c\sin x - c\,x\cos x + 2d\cos x - d\,x\sin x\,.$$

Plug these into (N) to obtain

$$y_p'' - y_p' - 6\,y_p \;\equiv\; 250\,x\cos x\,,$$

$$\begin{aligned}&-a\cos x - b\sin x - 2c\sin x - c\,x\cos x + 2d\cos x - d\,x\sin x\\ &\quad -\,(\,-a\sin x + b\cos x + c\cos x - c\,x\sin x + d\sin x + d\,x\cos x\,)\\ &\quad -\,6\,(\,a\cos x + b\sin x + c\,x\cos x + d\,x\sin x\,) \qquad \equiv \qquad 250\,x\cos x\,.\end{aligned}$$

Collect like terms:

$$\begin{aligned}&(-a + 2d - b - c - 6a)\;\cos x + (-b - 2c + a - d - 6b)\;\sin x\\ &\quad (-c - d - 6c)\,x\cos x + (-d + c - 6d)\,x\sin x \qquad \equiv \qquad 250\,x\cos x\,.\end{aligned}$$

simplify:

$$\begin{aligned}&(-7a - b - c + 2d)\;\cos x + (a - 7b - 2c - d)\;\sin x\\ &\quad (-7c - d)\,x\cos x + (c - 7d)\,x\sin x \qquad \equiv \qquad 250\,x\cos x \;+\; 0\,x\sin x\,.\end{aligned}$$

Equate like terms:

$$\begin{aligned} x\cos x &: \quad -7c - d \;\equiv\; 250\\ x\sin x &: \quad c - 7d \;\equiv\; 0\\ \cos x &: \quad -7a - b - c + 2d \;\equiv\; 0\\ \sin x &: \quad a - 7b - 2c - d \;\equiv\; 0\end{aligned}$$

Litmus Test: Note that these terms are exactly those terms that were in the "List".
Solve these to obtain

$$a = 2, \qquad b = 11, \qquad c = -35, \qquad d = -5\,.$$

So by (P), a particular solution of (N) is

$$\begin{aligned} y_p &= a\cos x + b\sin x + c\,x\cos x + d\,x\sin x\\ &= 2\cos x + 11\sin x - 35\,x\cos x - 5\,x\sin x\,.\end{aligned}$$

STEP 3:
Then the general solution of the nonhomogeneous problem (N) is

$$\begin{aligned} y &= y_c + y_p\\ &= c_1\,e^{3x} + c_2\,e^{-2x} + 2\cos x + 11\sin x - 35\,x\cos x - 5\,x\sin x\,.\end{aligned}$$

It is a 2–parameter family of solutions of (N).

STEP 4:
Apply initial conditions to the general solution found in Step 3, **NOT** to solution (C).