Solve the nonhomogeneous ODE

$$y'' - y' - 6y \; = \; 18x^2 - 52\cos 2x\,. \qquad \textbf{(N)}$$

NOTE: The input function is $g(x) = 18x^2 - 52\cos 2x$.

STEP 1:
Solve the *associated homogeneous problem*:

$$y'' - y' - 6y \; = \; 0\,. \qquad \textbf{(H)}$$

We did this in Example 1, so the *complementary solution* of (N) is

$$y_c \; = \; c_1\, e^{3x} + c_2\, e^{-2x}\,. \qquad \textbf{(C)}$$

STEP 2:
Construct a *particular solution* y_p of (N) by **Undetermined Coefficients**.

Input Function:	Terms
$g \; = \; 18x^2 - 52\cos 2x$	$x^2,\ \cos 2x$
$g' \; = \; 36x + 104\sin 2x$	$x,\ \sin 2x$
$g'' \; = \; 36 + 208\cos 2x$	$a,\ \cos 2x$
List:	$a,\ x,\ x^2,\ \cos 2x,\ \sin 2x$

$\mathcal{Q}$: Do any terms in the List already appear in y_c?
$\mathcal{A}$: No, so we need not modify any terms in the List.

So we seek a particular solution of (N) that is a linear combination of terms in the List:

$$y_p \; = \; a + bx + cx^2 + d\cos 2x + e\sin 2x\,. \qquad \textbf{(P)}$$

We will substitute y_p into (N), but first

$$\begin{aligned} y_p &= a + bx + cx^2 + d\cos 2x + e\sin 2x\,, \\ y_p' &= b + 2cx - 2d\sin 2x + 2e\cos 2x\,, \\ y_p'' &= 2c - 4d\cos 2x - 4e\sin 2x\,. \end{aligned}$$

Plug these into (N) to obtain

$$y_p'' - y_p' - 6\,y_p \;\equiv\; 18x^2 - 52\cos 2x\,,$$

$$\begin{aligned} &2c - 4d\cos 2x - 4e\sin 2x \\ &\quad -\,(\,b + 2cx - 2d\sin 2x + 2e\cos 2x\,) \\ &\quad -\,6\,(\,a + bx + cx^2 + d\cos 2x + e\sin 2x\,) \quad\equiv\quad 18\,x^2 - 52\,\cos 2x\,. \end{aligned}$$

Collect like terms:

$$\begin{aligned} &(-4d - 2e - 6d)\,\cos 2x + (-4e + 2d - 6e)\,\sin 2x \\ &\quad -\,6c\,x^2 + (-2c - 6b)\,x + (2c - b - 6a) \quad\equiv\quad 18\,x^2 - 52\,\cos 2x\,. \end{aligned}$$

simplify:

$$\begin{aligned} &(-10d - 2e)\,\cos 2x + (2d - 10e)\,\sin 2x \\ &\quad -\,6c\,x^2 + (-2c - 6b)\,x + (2c - b - 6a) \quad\equiv\quad 0 + 0\,x + 18\,x^2 - 52\,\cos 2x + 0\,\sin 2x\,. \end{aligned}$$

Equate like terms:

$$\begin{array}{rcll} x^2 & : & -6c \;\equiv\; 18 & \Longrightarrow\quad c \;=\; -3 \\ x & : & -2c - 6b \;\equiv\; 0 & \Longrightarrow\quad b \;=\; (-1/3)\,c \;=\; 1 \\ k & : & 2c - b - 6a \;\equiv\; 0 & \Longrightarrow\quad a \;=\; (2c - b)/6 \;=\; -7/6 \\ \cos 2x & : & -10d - 2e \;\equiv\; -52 & \\ \sin 2x & : & 2d - 10e \;\equiv\; 0 & \end{array}$$

The latter two equations give $d = 5$ and $e = 1$.

Litmus Test: Note that these terms are exactly those terms that were in the "List".

So by (P), a particular solution of (N) is

$$\begin{aligned} y_p &= a + bx + cx^2 + d\cos 2x + e\sin 2x \\ &= -\frac{7}{6} + x - 3x^2 + 5\cos 2x + \sin 2x\,. \end{aligned}$$

STEP 3:
Then the general solution of the nonhomogeneous problem (N) is

$$\begin{aligned} y &= y_c + y_p \\ &= c_1\,e^{3x} + c_2\,e^{-2x} - \frac{7}{6} + x - 3x^2 + 5\cos 2x + \sin 2x\,. \end{aligned}$$

It is a 2–parameter family of solutions of (N).

STEP 4:
Apply initial conditions to the general solution found in Step 3, **NOT** to solution (C).