Solve the nonhomogeneous ODE

$$y'' - y' - 6y \;=\; 25xe^{3x}\,. \tag{N}$$

NOTE: The input function is $g(x) = 25xe^{3x}$.

STEP 1:
Solve the *associated homogeneous problem*:

$$y'' - y' - 6y \;=\; 0\,. \tag{H}$$

We did this in Example 1, so the *complementary solution* of (N) is

$$y_c \;=\; c_1\,e^{3x} + c_2\,e^{-2x}\,. \tag{C}$$

STEP 2:
Construct a *particular solution* y_p of (N) by **Undetermined Coefficients**.

Input Function:	Terms
$g \;=\; 25xe^{3x}$	xe^{3x}
$g' \;=\; 25e^{3x} + 75xe^{3x}$	$e^{3x},\; xe^{3x}$
List:	$e^{3x},\; xe^{3x}$
New List:	$xe^{3x},\; x^2e^{3x}$

$\mathcal{Q}$: Do any terms in the List already appear in y_c?
$\mathcal{A}$: Yes, so we must modify the repeating terms:

Term e^{3x} appears in y_c, so e^{3x} repeats once in the List. So attach an x^1 to it to get xe^{3x}.
But term xe^{3x} already appears in the List, so attach another x^1 to it. So we get

$$\begin{aligned} e^{3x} &\;\to\; xe^{3x} \\ xe^{3x} &\;\to\; x^2e^{3x} \end{aligned}$$

So a particular solution y_p of (N) must be a linear combination of terms in the New List:

$$y_p \;=\; axe^{3x} + bx^2e^{3x}\,. \tag{P}$$

We will substitute y_p into (N), but first

$$\begin{aligned} y_p &= axe^{3x} + bx^2e^{3x}\,, \\ y_p' &= ae^{3x} + 3axe^{3x} + 2bxe^{3x} + 3bx^2e^{3x}\,, \\ y_p'' &= 6ae^{3x} + 9axe^{3x} + 2be^{3x} + 12bxe^{3x} + 9bx^2e^{3x}\,. \end{aligned}$$

Plug these into (N) to obtain

$$y_p'' - y_p' - 6\,y_p \;\equiv\; 25xe^{3x}\,,$$

$$\begin{aligned} &6a\,e^{3x} + 9a\,xe^{3x} + 2b\,e^{3x} + 12b\,xe^{3x} + 9b\,x^2e^{3x} \\ &\quad -(\;a\,e^{3x} + 3a\,xe^{3x} + 2b\,xe^{3x} + 3b\,x^2e^{3x}\,) \\ &\quad -6\,(\,a\,xe^{3x} + b\,x^2e^{3x}\,) \qquad \equiv \qquad 25\,xe^{3x}\,. \end{aligned}$$

Collect like terms:

$$(9b - 3b - 6b)\,x^2e^{3x} + (9a + 12b - 3a - 2b - 6a)\,xe^{3x} + (6a + 2b - a)\,e^{3x} \;\;\equiv\;\; 25xe^{3x}\,,$$

simplify:

$$10b\,xe^{3x} + (5a + 2b)\,e^{3x} \;\;\equiv\;\; 0\,e^{3x} \;+\; 25\,xe^{3x}\,.$$

Equate like terms:

$$\begin{aligned} xe^{3x} &: \quad 10b \;\equiv\; 25 \qquad \Longrightarrow \quad b \;=\; 5/2\;, \\ e^{3x} &: \quad 5a + 2b \;\equiv\; 0 \quad \Longrightarrow \quad a \;=\; (-2/5)\,b \;=\; -1\;. \end{aligned}$$

Litmus Test: Note that these terms are exactly those terms that were in the "List".

So by (P), a particular solution of (N) is

$$y_p \;=\; axe^{3x} + bx^2e^{3x} \;=\; -xe^{3x} + \frac{5}{2}x^2e^{3x}\,.$$

STEP 3:
Then the general solution of the nonhomogeneous problem (N) is

$$\begin{aligned} y &= y_c + y_p \\ &= c_1\,e^{3x} + c_2\,e^{-2x} - xe^{3x} + \frac{5}{2}x^2e^{3x}\,. \end{aligned}$$

It is a 2–parameter family of solutions of (N).

STEP 4:
Apply initial conditions to the general solution found in Step 3, **NOT** to solution (C).