Solve the nonhomogeneous ODE

$$y'' - y' - 6y \;=\; 25\,e^{3x}\,. \qquad \textbf{(N)}$$

NOTE: The input function is $g(x) = 25\,e^{3x}$.

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**STEP 1:**
Solve the *associated homogeneous problem*:

$$y'' - y' - 6y \;=\; 0\,. \qquad \textbf{(H)}$$

We did this in Example 1, so the *complementary solution* of (N) is

$$y_c \;=\; c_1\,e^{3x} + c_2\,e^{-2x}\,. \qquad \textbf{(C)}$$

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**STEP 2:**
Construct a *particular solution* $y_p$ of (N) by **Undetermined Coefficients**.

| Input Function: | Terms |
|---|---|
| $g \;=\; 25\,e^{3x}$ | $e^{3x}$ |
| $g' \;=\; 75\,e^{3x}$ | $e^{3x}$ |
| List: | $e^{3x}$ |
| New List: | $xe^{3x}$ |

$\mathcal{Q}$: Do any terms in the List already appear in $y_c$?
$\mathcal{A}$: Yes, so we must modify the repeating terms:

Term $e^{3x}$ appears in $y_c$, so $e^{3x}$ repeats once in the List. So attach an $x^1$ to it to get $xe^{3x}$:

$$e^{3x} \;\rightarrow\; xe^{3x}\,.$$

So a particular solution $y_p$ of (N) must be a linear combination of terms in the New List:

$$y_p \;=\; axe^{3x}\,. \qquad \textbf{(P)}$$

We will substitute $y_p$ into (N), but first

$$\begin{aligned} y_p &= axe^{3x}\,, \\ y_p' &= ae^{3x} + 3axe^{3x}\,, \\ y_p'' &= 6ae^{3x} + 9axe^{3x}\,. \end{aligned}$$

Plug these into (N) to obtain

$$y_p'' - y_p' - 6\,y_p \;\equiv\; 25\,e^{3x}\,,$$

$$6a\,e^{3x} + 9a\,xe^{3x} - (\,a\,e^{3x} + 3a\,xe^{3x}\,) - 6a\,xe^{3x} \quad\equiv\quad 25\,e^{3x}\,.$$

Collect like terms:

$$(9a - 3a - 6a)\,xe^{3x} + (6a - a)\,e^{3x} \;\equiv\; 25\,e^{3x}\,.$$

simplify:

$$5a\,e^{3x} \quad\equiv\quad 25\,e^{3x}\,.$$

Equate like terms:

$$e^{3x} \;\;:\;\; 5a \;\equiv\; 25 \quad\Longrightarrow\quad a \;=\; 5$$

**Litmus Test:** Note that this term is exactly that term that was in the “List”.

So by (P), a particular solution of (N) is

$$y_p \;=\; axe^{3x} \;=\; 5xe^{3x}\,.$$

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**STEP 3:**
Then the general solution of the nonhomogeneous problem (N) is

$$\begin{aligned} y &= y_c + y_p \\ &= c_1\,e^{3x} + c_2\,e^{-2x} + 5xe^{3x}\,. \end{aligned}$$

It is a 2–parameter family of solutions of (N).

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**STEP 4:**
Apply initial conditions to the general solution found in Step 3, **NOT** to solution (C).