Solve the nonhomogeneous ODE

$$y'' - y' - 6y \;=\; -x\,e^{4x}\,. \qquad \textbf{(N)}$$

NOTE: The input function is $g(x) = -x\,e^{4x}$.

STEP 1:
Solve the *associated homogeneous problem*:

$$y'' - y' - 6y \;=\; 0\,. \qquad \textbf{(H)}$$

We did this in Example 1, so the *complementary solution* of (N) is

$$y_c \;=\; c_1\,e^{3x} + c_2\,e^{-2x}\,. \qquad \textbf{(C)}$$

STEP 2:
Construct a *particular solution* y_p of (N) by **Undetermined Coefficients**.

Input Function:	Terms
$g \;=\; -xe^{4x}$	$x\,e^{4x}$
$g' \;=\; -e^{4x} - 4xe^{4x}$	$e^{4x},\ xe^{4x}$
$g'' \;=\; -8e^{4x} - 16xe^{4x}$	$e^{4x},\ xe^{4x}$
List:	$e^{4x},\ xe^{4x}$

$\mathcal{Q}$: Do any terms in the list already appear in y_c?
$\mathcal{A}$: No, so we need not modify any terms in the List.

So a particular solution y_p of (N) must be a linear combination of terms in the List:

$$y_p \;=\; ae^{4x} + bxe^{4x}\,. \qquad \textbf{(P)}$$

The coefficients a and b are to be determined.

We will substitute y_p into (N), but first

$$\begin{aligned} y_p &= ae^{4x} + bxe^{4x}\,, \\ y_p' &= 4ae^{4x} + be^{4x} + 4bxe^{4x}\,, \\ y_p'' &= 16ae^{4x} + 8be^{4x} + 16bxe^{4x}\,. \end{aligned}$$

Plug these into (N) to obtain

$$y_p'' - y_p' - 6\,y_p \;\equiv\; -xe^{4x}\,,$$

$$16ae^{4x} + 8b\,e^{4x} + 16b\,xe^{4x} - (\,4ae^{4x} + b\,e^{4x} + 4b\,xe^{4x}\,) - 6\,(\,ae^{4x} + b\,xe^{4x}\,) \;\equiv\; -\,xe^{4x}\,.$$

Collect like terms:

$$(\,16b - 4b - 6b\,)\,xe^{4x} + (\,16a + 8b - 4a - b - 6a\,)\,e^{4x} \;\equiv\; -x\,e^{4x}\,,$$

simplify:

$$6b\,xe^{4x} + (\,6a + 7b\,)\,e^{4x} \;\equiv\; 0\,e^{4x} -\, xe^{4x}\,.$$

Equate like terms:

$$\begin{aligned} xe^{4x} &: \quad 6b \,\equiv\, -1 & \Longrightarrow \quad & b \,=\, -1/6 \\ e^{4x} &: \quad 6a + 7b \,\equiv\, 0 & \Longrightarrow \quad & a \,=\, (-7/6)\,b \,=\, 7/36 \end{aligned}$$

Litmus Test: Note that these terms are exactly those terms that were in the "List".

So by (P), a particular solution of (N) is

$$y_p \;=\; ae^{4x} + bxe^{4x} \;=\; \frac{7}{36}\,e^{4x} - \frac{1}{6}\,xe^{4x}\,.$$

STEP 3:
Then the general solution of the nonhomogeneous problem (N) is

$$\begin{aligned} y &= y_c + y_p \\ &= c_1\,e^{3x} + c_2\,e^{-2x} + \frac{7}{36}\,e^{4x} - \frac{1}{6}\,xe^{4x}\,. \end{aligned}$$

It is a 2–parameter family of solutions of (N).

STEP 4:
Apply initial conditions to the general solution found in Step 3, **NOT** to solution (C).