Solve the nonhomogeneous ODE

$$y'' - y' - 6y \;=\; 18\,x^2\,. \qquad \textbf{(N)}$$

NOTE: The input function is $g(x) = 18x^2$.

STEP 1:
Solve the *associated homogeneous problem*:

$$y'' - y' - 6y \;=\; 0\,. \qquad \textbf{(H)}$$

We did this in Example 1, so the *complementary solution* of (N) is

$$y_c \;=\; c_1\,e^{3x} + c_2\,e^{-2x}\,. \qquad \textbf{(C)}$$

STEP 2:
Construct a *particular solution* y_p of (N) by **Undetermined Coefficients**.

Input Function:	Terms
$g \;=\; 18x^2$	x^2
$g' \;=\; 36x$	x
$g'' \;=\; 36$	a (const)
List:	a, x, x^2

$\mathcal{Q}$: Do any terms in the list already appear in y_c?
$\mathcal{A}$: No, so we need not modify any terms in the List.

So a particular solution y_p of (N) must be a linear combination of terms in the List:

$$y_p \;=\; a + bx + cx^2\,. \qquad \textbf{(P)}$$

The coefficients a, b, c are to be determined, hence the name "Undetermined Coefficients".

We will substitute y_p into (N), but first

$$\begin{aligned} y_p &= a + bx + cx^2\,, \\ y_p' &= b + 2cx\,, \\ y_p'' &= 2c\,. \end{aligned}$$

Plug these into (N) to obtain

$$y_p'' - y_p' - 6\,y_p \;\equiv\; 18\,x^2\,,$$

$$2c - (\,b + 2cx\,) - 6\,(\,a + bx + cx^2\,) \;\;\equiv\;\; 18\,x^2\,.$$

Collect like terms:

$$2c - b - 2cx - 6a - 6bx - 6cx^2 \equiv 18\,x^2\,,$$

simplify:

$$-6c\,x^2 + (-2c - 6b)\,x + (2c - b - 6a) \equiv 0 + 0\,x + 18\,x^2\,.$$

Equate like terms:

$$\begin{aligned}
&x^2\!: \quad -6c \equiv 18 && \Longrightarrow \quad c = -3 \\
&x\!: \quad -2c - 6b \equiv 0 && \Longrightarrow \quad b = -c/3 = 1 \\
&k\!: \quad 2c - b - 6a \equiv 0 && \Longrightarrow \quad a = (2c - b)/6 = -7/6
\end{aligned}$$

Litmus Test: Note that these terms are exactly those terms that were in the "List".

So by (P), a particular solution of (N) is

$$y_p = a + bx + cx^2 = -\frac{7}{6} + x - 3\,x^2\,.$$

STEP 3:
Then the general solution of the nonhomogeneous problem (N) is

$$\begin{aligned}
y &= y_c + y_p \\
&= c_1\,e^{3x} + c_2\,e^{-2x} - \frac{7}{6} + x - 3\,x^2\,.
\end{aligned}$$

It is a 2–parameter family of solutions of (N).

STEP 4:
Apply initial conditions to the general solution found in Step 3, **NOT** to solution (C).