Suppose a linear, homogeneous ODE with constant coefficients has a characteristic equation with these roots:

$$m \;=\; 0, \quad 0, \quad 0, \quad 0, \quad 3 \pm 5i, \quad 3 \pm 5i, \quad 3 \pm 5i\,.$$

Write the general solution of the ODE.

SOLUTION:

Since the characteristic equation has 10 roots, there must be 10 linearly independent solutions. They are:

$$m_1 = 0 \;\;\rightarrow\;\; y_1 = e^{0x} \;=\; 1 \qquad \text{(appears once)}\,,$$

$$m_2 = 0 \;\;\rightarrow\;\; y_2 = xe^{0x} \;=\; x \qquad \text{(repeats once)}\,,$$

$$m_3 = 0 \;\;\rightarrow\;\; y_3 = x^2e^{0x} \;=\; x^2 \qquad \text{(repeats twice)}\,,$$

$$m_4 = 0 \;\;\rightarrow\;\; y_4 = x^3e^{0x} \;=\; x^3 \qquad \text{(repeats thrice)}\,,$$

$$m_{5,6} = 3 \pm 5i \;\;\rightarrow\;\; \begin{cases} y_5 = e^{3x}\cos 5x & \text{(appears once)} \\ y_6 = e^{3x}\sin 5x & \end{cases},$$

$$m_{7,8} = 3 \pm 5i \;\;\rightarrow\;\; \begin{cases} y_7 = xe^{3x}\cos 5x & \text{(repeats once)} \\ y_8 = xe^{3x}\sin 5x & \end{cases},$$

$$m_{9,10} = 3 \pm 5i \;\;\rightarrow\;\; \begin{cases} y_9 = x^2e^{3x}\cos 5x & \text{(repeats twice)} \\ y_{10} = x^2e^{3x}\sin 5x & \end{cases}.$$

So the general solution of the ODE is

$$\begin{aligned} y \;&=\; c_1\,y_1 + c_2\,y_2 + c_3\,y_3 + c_4\,y_4 + c_5\,y_5 + c_6\,y_6 + c_7\,y_7 + c_8\,y_8 + c_9\,y_9 + c_{10}\,y_{10} \\ &=\; c_1 + c_2x + c_3x^2 + c_4x^3 + c_5e^{3x}\cos 5x + c_6e^{3x}\sin 5x \\ &\qquad + c_7xe^{3x}\cos 5x + c_8xe^{3x}\sin 5x + c_9x^2e^{3x}\cos 5x + c_{10}x^2e^{3x}\sin 5x\,. \end{aligned}$$

This is a 10–parameter family of solutions. The parent ODE would have been 10th order. The characteristic equation would have been 10th degree.