Suppose a linear, homogeneous ODE with constant coefficients has a characteristic equation with these roots:

$$m = 0, \quad -1, \quad 3, \quad 3, \quad 3, \quad -4 \pm 6i\,.$$

Write the general solution of the ODE.

SOLUTION:

Since the characteristic equation has 7 roots, there must be 7 linearly independent solutions. They are:

$$m_1 = 0 \;\to\; y_1 = e^{0x} \;=\; 1\,,$$

$$m_2 = -1 \;\to\; y_2 = e^{-x}\,,$$

$$m_3 = 3 \;\to\; y_3 = e^{3x} \qquad \text{(appears once)}\,,$$

$$m_4 = 3 \;\to\; y_4 = xe^{3x} \qquad \text{(repeats once)}\,,$$

$$m_5 = 3 \;\to\; y_5 = x^2e^{3x} \qquad \text{(repeats twice)}\,,$$

$$m_{6,7} = -4 \pm 6i \;\to\; \begin{cases} y_6 = e^{-4x}\cos 6x \\ y_7 = e^{-4x}\sin 6x \end{cases}.$$

So the general solution of the ODE is

$$\begin{aligned} y &= c_1\,y_1 + c_2\,y_2 + c_3\,y_3 + c_4\,y_4 + c_5\,y_5 + c_6\,y_6 + c_7\,y_7 \\ &= c_1 + c_2e^{-x} + c_3e^{3x} + c_4xe^{3x} + c_5x^2e^{3x} + c_6e^{-4x}\cos 6x + c_7e^{-4x}\sin 6x\,. \end{aligned}$$

This is a 7–parameter family of solutions. The parent ODE would have been 7th order. The characteristic equation would have been 7th degree.