Solve the homogeneous ODE

$$y''' + y'' - 8y' - 12y \; = \; 0 \,. \qquad \textbf{(H)}$$

This ODE is 3rd order, so we must determine 3 linearly independent solutions.

Characteristic Equation:

$$\begin{aligned} & m^3 + m^2 - 8m - 12 \; = \; 0 \\ \Longrightarrow \quad & (m-3)(m+2)^2 \; = \; 0 \\ \Longrightarrow \quad & (m-3)(m+2)(m+2) \; = \; 0 \\ \Longrightarrow \quad & m_1 = +3, \quad m_2 = -2, \quad m_3 = -2 \,. \end{aligned}$$

Here the roots m_2 and m_3 are repeating. I say the root -2 *appears once* in m_2 and *repeats once* in m_3.

So 3 linearly independent solutions of (H) are

$$\begin{aligned} m_1 = +3 \;\; &\rightarrow \;\; y_1 = e^{3x} \,, \\ m_2 = -2 \;\; &\rightarrow \;\; y_2 = e^{-2x} \,, \qquad \text{(appears once)} \\ m_3 = -2 \;\; &\rightarrow \;\; y_3 = xe^{-2x} \qquad \text{(repeats once)} \,. \end{aligned}$$

So the general solution of (H) is

$$\begin{aligned} y \;\; &= \;\; c_1 \, y_1 + c_2 \, y_2 + c_3 \, y_3 \\ &= \;\; c_1 \, e^{3x} + c_2 \, e^{-2x} + c_3 \, x \, e^{-2x} \,. \end{aligned}$$

This is a 3–parameter family of solutions.