Solve the homogeneous ODE

$$y'' - 4y' + 13y = 0\,. \qquad \textbf{(H)}$$

Characteristic Equation:

$$m^2 - 4m + 13 = 0\,.$$

So

$$\begin{aligned} m_{1,2} &= \frac{4 \pm \sqrt{16 - 4\cdot 13}}{2} \\ &= \frac{4 \pm \sqrt{-36}}{2} \\ &= \frac{4 \pm 6i}{2} \\ &= 2 \pm 3i \qquad (\equiv\ \alpha \pm \beta i)\,. \end{aligned}$$

These 2 roots are complex conjugates with $\alpha = 2$ (real part) and $\beta = 3$ (imaginary part).

So 2 linearly independent solutions of (H) are

$$\begin{aligned} y_1 &= e^{\alpha x}\cos\beta x = e^{2x}\cos 3x\,, \\ y_2 &= e^{\alpha x}\sin\beta x = e^{2x}\sin 3x\,. \end{aligned}$$

So the general solution of (H) is

$$\begin{aligned} y &= c_1\,y_1 + c_2\,y_2 \\ &= c_1\,e^{2x}\cos 3x + c_2\,e^{2x}\sin 3x \qquad \text{or} \\ &= e^{2x}\,(c_1\cos 3x + c_2\sin 3x)\,. \end{aligned}$$

This is a 2–parameter family of solutions.