Example 3: Solve the ODE

$$(x+1)\,dy + (x+2)y\,dx = 2xe^{-x}dx\,. \qquad (0)$$

Write (0) in std linear form (1): $\div\, dx$ and $\div\,(x+1)$:

$$1\,y' + \left(\tfrac{x+2}{x+1}\right)y = \tfrac{2x}{x+1}e^{-x}. \qquad (1)$$

So $P(x) = (x+2)/(x+1)$.

$$e^{a+b} = e^a \cdot e^b$$

1. Set

$$r = \int P(x)\,dx = \int \frac{x+2}{x+1}\,dx$$

let $u = x+1$

$$= \cdots = x + \ln(x+1)\,.$$

2. Set

$$\mu \equiv e^r = e^{x+\ln(x+1)} = e^x \cdot e^{\ln(x+1)} = (x+1)e^x. \quad (2)$$

This is the **integrating factor** of (1).

$$y' + \left(\tfrac{x+2}{x+1}\right)y = \tfrac{2x}{x+1}e^{-x} \qquad (1)$$

3. Mult. ODE (1) by $\mu = (x+1)e^x$: $\quad (e^x \cdot e^{-x} = 1)$

$$(x+1)e^x y' + (x+2)e^x y = 2x\,. \qquad (3)$$

Litmus Test: The LHS(3) **must** be $\frac{d}{dx}(\mu y)$.

Check: You show

$$\frac{d}{dx}(\mu y) = \frac{d}{dx}\Big[(x+1)e^x y\Big] \stackrel{?}{=} \cdots = \text{LHS(3)}\,. ✓$$

So ODE (3) simplifies to

$$\frac{d}{dx}\Big[(x+1)e^x y\Big] = 2x\,. \qquad (4)$$

$$\frac{d}{dx}\Big[(x+1)e^{x}y\Big] \;=\; 2x\,. \qquad (4)$$

4. Integrate both sides of (4) wrt x:

$$\int \frac{d}{dx}\Big[(x+1)e^{x}y\Big]\,dx \;=\; \int 2x\,dx$$

$$\Longrightarrow \qquad (x+1)e^{x}y \;=\; x^2 + c \qquad \textbf{Implicit form}$$

$$\Longrightarrow \qquad \boxed{y \;=\; \frac{x^2+c}{x+1}\,e^{-x}.} \qquad \textbf{Explicit form} \quad x \neq -1$$

5. Apply any given IC to determine c.

Comments:

- **Always** write the soln of a linear ODE in explicit form.
- **Integrating Factors** works only for 1st order **linear** ODEs.