THIS IS NOT INTENDED TO BE A SAMPLE FINAL EXAM.

1. For each differential equation, state its order and circle whether it is linear or nonlinear. **For each linear problem**, also state whether it is **homogeneous** or **nonhomogeneous**. In each case, y is a function of x.

Equation	Order	(Circle One)	If Linear, is it homogeneous?
$x\,dy = (y + \cos x)\,dx$		linear / nonlinear	Yes / No
$4y''' + 2\,y = \ln y$		linear / nonlinear	Yes / No
$e^x\,y'' + x^2\,y^{(5)} - 3x = 0$		linear / nonlinear	Yes / No

2. Determine the Wronskian of the following set of functions and simplify. Then answer the question below.

$$f_1(x) = e^{3x}, \qquad f_2(x) = 4e^{3x}$$

Based on the Wronskian, are functions f_1 and f_2 **linearly dependent** or **linearly independent**?

3. A certain homogeneous, linear ODE has a characteristic equation with roots

$$-3, \quad -3, \quad -2 \pm 5i, \quad \pm i, \quad \pm i$$

   (a) Write the general solution.
   (b) What is the order of the ODE?

4. Obtain the general solution of each ODE. **Answer:**

   (a) $y^{(5)} + 3y^{(4)} - 18y''' = 0$ $\qquad y = c_1 + c_2x + c_3x^2 + c_4e^{3x} + c_5e^{-6x}$
   (b) $y^{(4)} + 2y''' - 8y'' = 0$ $\qquad y = c_1 + c_2x + c_3e^{-4x} + c_4e^{2x}$
   (c) $y'' + 8y' + 20y = 0$ $\qquad y = c_1e^{-4x}\cos 2x + c_2e^{-4x}\sin 2x$
   (d) $y^{(5)} - 12y^{(4)} + 40y''' = 0$ $\qquad y = c_1 + c_2x + c_3x^2 + c_4e^{6x}\cos 2x + c_5e^{6x}\sin 2x$

5. The following pertain to a spring–mass apparatus or to an RLC circuit. Circle true or false.

   - **TRUE / FALSE:** The natural frequency is determined by the driving force (voltage source).
   - **TRUE / FALSE:** Pure resonance cannot exist when damping is present.
   - **TRUE / FALSE:** Changing the initial conditions changes the steady state term.
   - **TRUE / FALSE:** A transient term decays to zero as time becomes large.
   - **TRUE / FALSE:** The particular solution determines the natural frequency.
   - **TRUE / FALSE:** As time increases, we may approximate the soln by the complementary soln.

6. Solve the following by **separating variables**. **Answer:**

   (a) $4\,(xy + 2y)\,dy + xe^{y^2}\,dx = 0, \quad y(-3) = 0$ $\qquad 2e^{-y^2} = x - 2\ln|x+2| + 5$
   (b) $(x + 3xy^2)\,dy + 2(1 - x^2)\,dx = 0, \quad y(-1) = 2$ $\qquad y + y^3 = x^2 - 2\ln|x| + 9$
   (c) $y\,\cos^2 x\,dy - (y^2 - 1)\,dx = 0, \quad y(\pi/4) = 0$ $\qquad \ln|y^2 - 1| = 2\tan x - 2$

7. Verify that the given equation is **homogeneous** and state its degree. Then solve it by using an appropriate substitution. **Answer:**

   (a) $x\,y^2\,dy = (y^3 - x^3)\,dx\,, \qquad y(1) = 2$ $\qquad\qquad y^3 = x^3\Big(8 - 3\ln|x|\Big)$

   (b) $x^2\,dy - (xy + y^2)\,dx = 0\,, \qquad y(1) = 4$ $\qquad\qquad y = 4x/(1 - 4\ln|x|)$

8. Verify that the following ODE is **exact**, and then solve it accordingly. **Answer:**

   (a) $(6\,x\,y^3 + \cos y)\,dx + (9\,x^2\,y^2 - x\,\sin y)\,dy = 0$ $\qquad\qquad 3x^2y^3 + x\cos y = c$

   (b) $(4x^3y^2 + 3/x)\,dx + (2x^4y + \sin y)\,dy = 0$ $\qquad\qquad x^4\,y^2 + 3\ln|x| - \cos y = c$

9. Solve the **linear** ODE using the appropriate method. Write the solution y **explicitly** in terms of x. **Answer:**

   (a) $x^2\,\dfrac{dy}{dx} + (x + 2\,x^3)\,y = 6\,x^2$ $\qquad\qquad y = \dfrac{3 + c\,e^{-x^2}}{x}$

   (b) $x\,\dfrac{dy}{dx} + (3\,x + 1)\,y = 8\,x^3\,e^{-3x}$ $\qquad\qquad y = \dfrac{2\,x^4 + c}{xe^{3x}}$

10. A 3 kg mass is attached to a spring with spring constant 48 N/m and immersed in a fluid that offers a resistance force numerically equal to β times the instantaneous velocity. State the range of values of β for which the resulting motion is **underdamped**. **Answer:** $\beta < 24$

11. A circuit comprises a 2 h inductor, 1/50 f capacitor, a 16 Ω resistor, and a voltage source of $(240\cos 5t)$ V. Initially there is a 6 C charge on the capacitor and a -3 amp current in the circuit. Determine the charge stored on the capacitor a time t. Also determine the steady state charge, the steady state amplitude, and the steady state frequency. **Answer:** $q(t) = e^{-4t}(6\cos 3t + 2\sin 3t) + 3\sin 5t$, 3C, $5/(2\pi)$ Hz

12. A 2 kg mass is attached to a spring with constant 50 N/m and immersed in a fluid which offers a resistance force numerically equal to 12 times the instantaneous velocity. Initially the mass is given a downward velocity of 1 m/sec from 3 m above the equilibrium position. Obtain the equation of motion $x(t)$, and state whether it is overdamped, underdamped, or critically damped. **Answer:** $x(t) = -e^{-3t}(3\cos 4t + 2\sin 4t)$, UD

13. Suppose an RLC circuit has solution

$$q(t) = 8\,\cos 4t - 6\,\sin 4t + e^{-3t}(-4\,\cos 2t + 3\,\sin 2t)\,.$$

   - What is the steady state charge?
   - What is the amplitude of the steady state charge?
   - What is the frequency of the steady state charge? (include units)
   - What is the steady state current?

14. **[6 pts]** Identify each function as overdamped, underdamped, critically damped, or simple harmonic.

   - $x(t) = e^{-3t}(2\cos 4t - 5\sin 4t)$
   - $x(t) = 2\cos 4t - 5\sin 4t$
   - $x(t) = e^{-3t}(2 - 5t)$

15. An electrical circuit comprises a 2 h inductor, a 1/18 f capacitor, and is subject to an external voltage source of $72\cos 3t$ V. There is no resistor.

   (a) Write the differential equation governing the charge $q(t)$. Do **NOT** solve it.

   (b) Determine the complementary solution. Do **NOT** determine the general solution.

   (c) What is the natural frequency?

   (d) Will the circuit undergo pure resonance? Briefly explain.

   (e) Now solve the ODE to obtain the charge $q(t)$ to confirm your answer.

16. Use **undetermined coefficients** to obtain the general solution of the ODE. **Answer:**

(a) $y'' - 2y' - 8y \;=\; 18\,e^{4x} + 64\,x^2$ $\qquad y \;=\; c_1e^{4x} + c_2e^{-2x} - 3 + 4x - 8x^2 + 3xe^{4x}$

(b) $y'' - 9\,y \;=\; 10\,\cos x + 12\,e^{-3x}$ $\qquad y \;=\; c_1e^{3x} + c_2e^{-3x} - \cos x - 2xe^{-3x}$

17. Use **variation of parameters** to obtain **and simplify** the general solution of the ODE. **Answer:**

(a) $y'' - y' - 2\,y \;=\; 9\,e^{2x}$ $\qquad y \;=\; c_1e^{2x} + c_2e^{-x} + 3xe^{2x}$

(b) $y'' - 3\,y' + 2\,y \;=\; 8 + e^{2x}$ $\qquad y \;=\; c_1e^{2x} + c_2e^{x} + 4 + xe^{2x}$

18. Determine the following: **Answer:**

(a) $\mathcal{L}\Big\{e^{2t} + 6\,\sin 5t\Big\}$ $\qquad \dfrac{1}{s-2} + \dfrac{30}{s^2+25}$

(b) $\mathcal{L}\Big\{t^3 - \cosh 5t\Big\}$ $\qquad \dfrac{6}{s^4} - \dfrac{s}{s^2-25}$

(c) $\mathcal{L}\Big\{\sin^2 5t\Big\}$ $\qquad \dfrac{1}{2}\left(\dfrac{1}{s} - \dfrac{s}{s^2+100}\right)$

(d) $\mathcal{L}\Big\{\sin 5t\cos 5t\Big\}$ $\qquad \dfrac{5}{s^2+100}$

(e) $\mathcal{L}\Big\{e^{2t}\,\sin 5t\Big\}$ $\qquad \dfrac{5}{(s-2)^2+25}$

(f) $\mathcal{L}\Big\{e^{3t}\,\cos 5t\Big\}$ $\qquad \dfrac{s-3}{(s-3)^2+25}$

(g) $\mathcal{L}\Big\{e^{4t}\,\cosh 5t\Big\}$ $\qquad \dfrac{s-4}{(s-4)^2-25}$

(h) $\mathcal{L}\Big\{e^{t}\,\sinh 5t\Big\}$ $\qquad \dfrac{5}{(s-1)^2-25}$

(i) $\mathcal{L}\Big\{t^5\,e^{6t}\Big\}$ $\qquad \dfrac{120}{(s-6)^6}$

(j) $\mathcal{L}\Big\{t^2\,\mathcal{U}(t-5)\Big\}$ $\qquad e^{-5s}\left(\dfrac{2}{s^3} + \dfrac{10}{s^2} + \dfrac{25}{s}\right)$

(k) $\mathcal{L}\Big\{t^2\,e^{3t}\,\mathcal{U}(t-5)\Big\}$ $\qquad e^{-5(s-3)}\left[\dfrac{2}{(s-3)^3} + \dfrac{10}{(s-3)^2} + \dfrac{25}{s-3}\right]$

(l) $\mathcal{L}\Big\{\displaystyle\int_0^t (t-\tau)^4\,\cos 5\tau\,d\tau\Big\}$ $\qquad \dfrac{24}{s^4\,(s^2+25)}$

(m) $\mathcal{L}\Big\{\displaystyle\int_0^t \tau^3\,\sinh[5(t-\tau)]\,d\tau\Big\}$ $\qquad \dfrac{30}{s^4\,(s^2-25)}$

(n) $\mathcal{L}\Big\{\displaystyle\int_0^t e^{2(\tau-t)}\,\cos 5\tau\,d\tau\Big\}$ $\qquad \dfrac{s}{(s+2)\,(s^2+25)}$

(o) $\mathcal{L}\Big\{\displaystyle\int_0^t e^{2\tau}\,\cos 5\tau\,d\tau\Big\}$ $\qquad \dfrac{s-2}{s\,[(s-2)^2+25]}$

(p) $\mathcal{L}^{-1}\left\{\dfrac{1}{2s+8}\right\}$ $\qquad \dfrac{1}{2}\,e^{-4t}$

(q) $\mathcal{L}^{-1}\left\{\dfrac{7}{s^2+9}\right\}$ $\qquad \dfrac{7}{3}\,\sin 3t$

(r) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-16}\right\}$ $\qquad \cosh 4t$

(s) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s-5)^4}\right\}$ $\qquad e^{5t}\left(\dfrac{1}{2}\,t^2 + \dfrac{5}{6}\,t^3\right)$

(t) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s-5)^2+16}\right\}$ $\qquad e^{5t}\left(\cos 4t + \dfrac{5}{4}\sin 4t\right)$

(u) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-10s+41}\right\}$ $\qquad e^{5t}\left(\cos 4t + \dfrac{5}{4}\sin 4t\right)$

(v) $\mathcal{L}^{-1}\left\{\dfrac{6s-2}{s^2+2\,s-3}\right\}$ $\qquad e^t + 5\,e^{-3t}$ or $e^{-t}\left(6\cosh 2t - 4\sinh 2t\right)$

(w) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+6\,s+25}\right\}$ $\qquad e^{-3t}\left(\cos 4t - \dfrac{3}{4}\sin 4t\right)$

(x) $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s^2+16}\right\}$ $\qquad \dfrac{1}{4}\mathcal{U}(t-3)\sin 4(t-3)$

(y) $\mathcal{L}^{-1}\left\{\dfrac{s\,e^{-3s}}{s^2+16}\right\}$ $\qquad \mathcal{U}(t-3)\cos 4(t-3)$

(z) $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{(s-5)^2+16}\right\}$ $\qquad \dfrac{1}{4}\mathcal{U}(t-3)\,e^{5(t-3)}\sin 4(t-3)$

(z.1) $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s^2-10\,s+41}\right\}$ $\qquad \dfrac{1}{4}\mathcal{U}(t-3)\,e^{5(t-3)}\sin 4(t-3)$

19. Use **Laplace transforms** to solve for $y(t)$ or $f(t)$:

(a) $y'' - 10\,y' + 25\,y = 24\,t^2\,e^{5t}$, $\quad y(0) = -2$, $\quad y'(0) = -10$ $\qquad y(t) = e^{5t}\left(2\,t^4 - 2\right)$

(b) $y'' - 10\,y' + 25\,y = 24\,t^2\,e^{5t}$, $\quad y(0) = -2$, $\quad y'(0) = 10$ $\qquad y(t) = e^{5t}\left(2\,t^4 + 20\,t - 2\right)$

(c) $y'' - 2\,y' + y = e^t$, $\quad y(0) = 3$, $\quad y'(0) = -4$ $\qquad y(t) = \dfrac{1}{2}e^t\left(t^2 - 14\,t + 6\right)$

(d) $y'' - 6\,y' + 13\,y = 0$, $\quad y(0) = 0$, $\quad y'(0) = -3$ $\qquad y(t) = -\dfrac{3}{2}e^{3t}\sin 2t$

(e) $y'' - 5\,y' + 6\,y = \mathcal{U}(t-1)$, $\quad y(0) = 0$, $\quad y'(0) = 1$

$$y(t) = e^{3t} - e^{2t} + \frac{1}{6}\mathcal{U}(t-1)\left[1 - 3\,e^{2(t-1)} + 2\,e^{3(t-1)}\right]$$

(f) $y'' - 2\,y' = 8\,e^{-2t}$, $\quad y(0) = 1$, $\quad y'(0) = -4$ $\qquad y(t) = e^{-2t} - e^{2t} + 1$

(g) $y'' + 16\,y = 1$, $\quad y(0) = 1$, $\quad y'(0) = 2$ $\qquad y(t) = \dfrac{1}{16}\left(15\cos 4t + 8\sin 4t + 1\right)$

(h) $y' - 4y = 8\,\mathcal{U}(t-3)$, $\quad y(0) = 2$ $\qquad y(t) = 2\,e^{4t} + 2\,\mathcal{U}(t-3)\left[e^{4(t-3)} - 1\right]$

(i) $y' - 4\,y = 3 - 4\displaystyle\int_0^t y(\tau)\,d\tau$, $\quad y(0) = -6$ $\qquad y(t) = -e^{2t}\,(6 + 9\,t)$

(j) $y' = \cos t + \displaystyle\int_0^t y(\tau)\cos(t-\tau)\,d\tau$, $\quad y(0) = 1$ $\qquad y(t) = \dfrac{1}{2}t^2 + t + 1$

(k) $f(t) = 8 - 15\displaystyle\int_0^t \sin 5\tau\,f(t-\tau)\,d\tau$ $\qquad f(t) = 2 + 6\cos 10t$

(l) $f(t) = 2\,e^{-5t} - 10\displaystyle\int_0^t e^{5\tau}\,f(t-\tau)\,d\tau$ $\qquad f(t) = (2 - 20t)\,e^{-5t}$

(m) $f(t) = 14\,e^{-5t} - 3\displaystyle\int_0^t e^{5\tau}\,f(t-\tau)\,d\tau$ $\qquad f(t) = 20\,e^{-5t} - 6\,e^{2t}$

20. Express $f(t)$ in terms of unit steps functions, and then determine $\mathcal{L}\{f(t)\}$.

$$f(t) = \begin{cases} t^2, & 0 \le t < 3 \\ -e^{5t}, & t \ge 3 \end{cases} \qquad \mathcal{L}\{f(t)\} = \frac{2}{s^3} - e^{-3s}\left(\frac{9}{s} + \frac{6}{s^2} + \frac{2}{s^3}\right) - \frac{e^{-3(s-5)}}{s-5}$$