Consider a first order ODE of the form

$$M\,dx + N\,dy \;=\; 0\,. \tag{1}$$

This ODE is called **exact** if

$$\frac{\partial M}{\partial y} \;=\; \frac{\partial N}{\partial x}\,. \tag{ET}$$

This is called the **exactness test**.

SOLVING AN EXACT ODE

You must first confirm that the ODE is exact. Then to solve it:

1. Set
$$\frac{\partial f}{\partial x} \;=\; M\,, \tag{2a}$$
$$\frac{\partial f}{\partial y} \;=\; N\,. \tag{2b}$$

2. Integrate Eq. $(2\mathrm{a})$ partially with respect to x (treat y as a const) to obtain
$$f \;=\; \text{ftn}(x,y) + g(y)\,. \tag{3}$$

3. Force this expression for f to satisfy Eq. $(2\mathrm{b})$.

   That is, determine $\dfrac{\partial f}{\partial y}$ from Eq. (3), and set it equal to N.

   This will give an equation for $g'(y)$ (all x's should drop from the equation):
$$g'(y) \;=\; \text{ftn}(y)\,. \qquad \text{(There should be no } x\text{'s!)}$$

   Integrate this with respect to y to obtain $g(y)$, and subst. the result into Eq. (3) to obtain f.

4. Then the solution of ODE (1) is the equation
$$f \;=\; c\,, \tag{4}$$
   where c is an arbitrary constant. This solution is usually in **implicit** form. Simplify it to the extent possible. It is not always possible to express the solution in **explicit** form.

5. If there is an initial condition $y(x_0) = y_0$, apply it to (4) (that is, plug in x_0 for x and y_0 for y) to determine the constant c. Then plug that value of c into (4) to obtain the solution of the IVP.

NOTE: We do **NOT** integrate **both** equations $(2\mathrm{a})$ and $(2\mathrm{b})$. We integrate **only** $(2\mathrm{a})$.
