Example 3

A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. The density ρ [gm/cm^3] at any pt (x, y, z) in E is prop to its dist from the cylinder axis.

x, y, z are measured in cm.

Determine the mass of E.

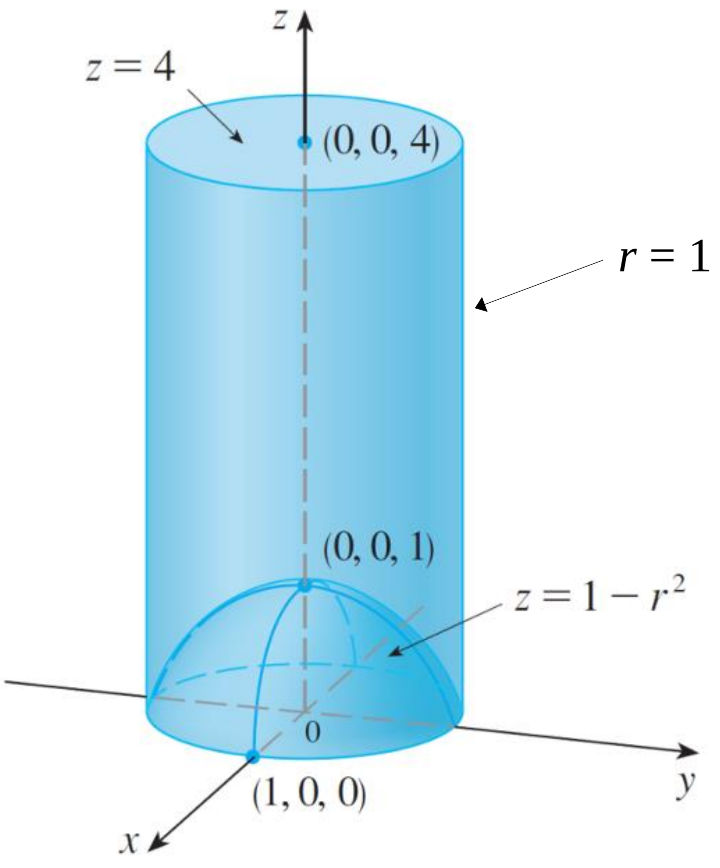


Figure 8

Example 3 – *Solution*

cont'd

Solution:

In cyl coords, $x^2 + y^2 = r^2$, so the cylinder is $r = 1$, and the paraboloid is $z = 1 - r^2$, so

$$E = \{ (r, \theta, z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ 1 - r^2 \le z \le 4 \}$$

Since the density at (x, y, z) is prop to the dist from the z-axis, the density ftn is

$$\rho = Kr$$

where K is the proportionality const.

Example 3 – *Solution*

cont'd

Therefore, the mass of E is

$$m = \underbrace{\iiint_E \rho\, dV}_{dm} = \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 \underbrace{(Kr)}_{\rho}\, \underbrace{r\, dz\, dr\, d\theta}_{dV}$$

$dm \longrightarrow \rho \cdot dV$

$$= \int_0^{2\pi}\int_0^1 Kr^2[4 - (1 - r^2)]\, dr\, d\theta$$

$$= K\int_0^{2\pi} d\theta \int_0^1 (3r^2 + r^4)\, dr$$

$$= 2\pi K\left[r^3 + \frac{r^5}{5}\right]_0^1 = \frac{12\pi K}{5} \quad \text{gm}$$

Example 4

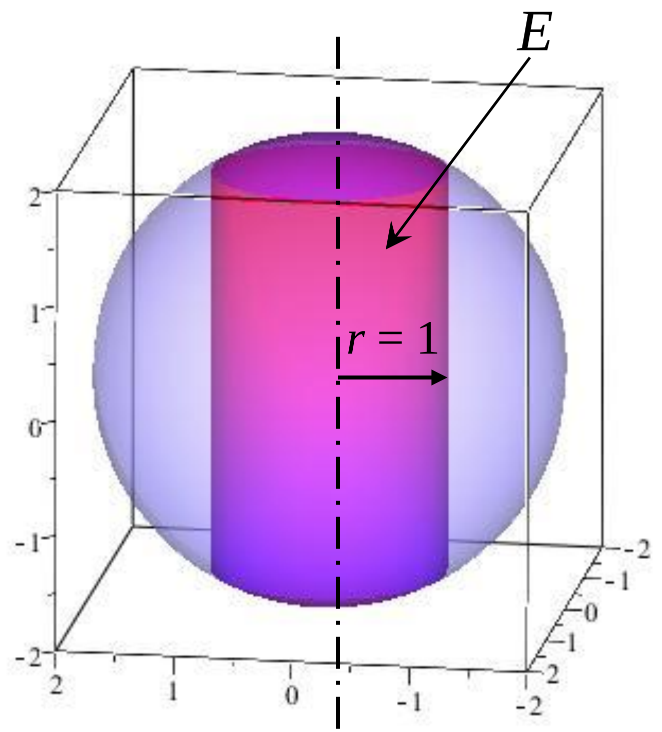
A solid E lies inside the cylinder $x^2 + y^2 = 1$ and inside the sphere of radius 2 centered at O. Determine the volume of E.

In cyl coords, the cylinder is given by $r = 1$, and the sphere by $x^2 + y^2 + z^2 = 4$ or $r^2 + z^2 = 4$.

We'll determine the volume of the upper half E_2 and multiply by 2.

So the limits of integration are:

	LL	UL
$r:$	$r = 0,$	$r = 1,$
$\theta:$	$\theta = 0,$	$\theta = 2\pi,$
$z:$	$z = 0,$	$z = \sqrt{4 - r^2}.$



Example 4 – *Solution*

cont'd

So the volume of E is

$$V(E) = 2\iiint_{E_2} \boxed{dV} = 2\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{\sqrt{4-r^2}} \overbrace{r\,dz\,dr\,d\theta}^{dV}$$

$$= 2\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\,z\Big|_{z=0}^{\sqrt{4-r^2}} dr\,d\theta$$

$$= 2\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\sqrt{4-r^2}\,dr\,d\theta \quad \left(\text{let } u = 4-r^2\right)$$

$$= 2\int_{\theta=0}^{2\pi} -\frac{1}{3}\left(4-r^2\right)^{3/2}\Big|_{r=0}^{1} d\theta$$

$$= -\frac{2}{3}\int_{\theta=0}^{2\pi}\left(3^{3/2}-4^{3/2}\right)d\theta$$

Example 4 – *Solution*

cont'd

$$V(E) = -\frac{2}{3}\int_{\theta=0}^{2\pi}\left(3^{3/2}-4^{3/2}\right)d\theta$$

$$= -\frac{2}{3}\left(3^{3/2}-4^{3/2}\right)\Big|_{\theta=0}^{2\pi}$$

$$= \frac{4\pi}{3}\left(8-3\sqrt{3}\right)$$

$$= 11.7447 \text{ units}^3.$$