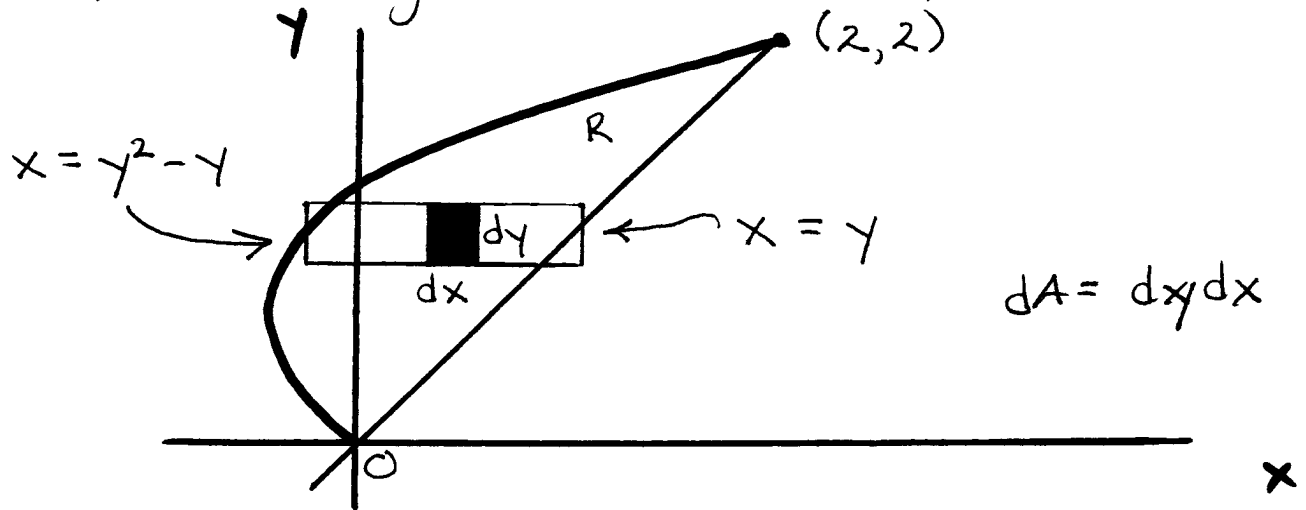


Ex. Find the volume of the solid under the paraboloid $z = 3x^2 + y^2$ and above the region bound by $y = x$ and $x = y^2 - y$.

Graph the region R in the xy -plane:



Outer rectangle:

LL: $y = 0$ thickness = dy

UL: $y = 2$

Inner rectangle:

LL: $x = y^2 - y$ thickness = dx

UL: $x = y$

$$z = f(x, y) = 3x^2 + y^2 \quad (\text{top of solid})$$

$$\begin{aligned} V &= \iint_R f(x, y) \, dA \\ &= \int_0^2 \int_{y^2 - y}^y (3x^2 + y^2) \, dy \, dx \end{aligned}$$

So

$$V = \int_0^2 \int_{y^2-y}^y (3x^2 + y^2) dx dy$$

$$= \int_0^2 (x^3 + y^2 x) \Big|_{x=y^2-y}^y dy$$

$$= \int_0^2 (x^3 \Big|_{x=y^2-y}^y + y^2 x \Big|_{x=y^2-y}^y) dy$$

$$= \int_0^2 \{ y^3 - (y^2-y)^3 + y^2(y - (y^2-y)) \} dy$$

$$= \int_0^2 \{ y^3 - (y^6 - 3y^5 + 3y^4 - y^3) + y^3 - y^4 + y^3 \} dy$$

$$= \int_0^2 [4y^3 - y^6 + 3y^5 - 4y^4] dy$$

$$= (y^4 - \frac{1}{7}y^7 + \frac{1}{2}y^6 - \frac{4}{5}y^5) \Big|_{y=0}^2$$

$$= y^4 \left(1 - \frac{1}{7}y^3 + \frac{1}{2}y^2 - \frac{4}{5}y \right) \Big|_{y=0}^2$$

$$= 16 \left(1 - \frac{8}{7} + 2 - \frac{8}{5} \right)$$

$$= 16 \left(\frac{9}{35} \right)$$

$$= \frac{144}{35} \text{ units}^3$$