Local Extrema and Saddle Points (Second Derivative Test)

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$restart$
$with(plots):\quad with(plottools):$
$a := -2$

$$-2 \tag{1}$$

$b := 2$

$$2 \tag{2}$$

$c := -3$

$$-3 \tag{3}$$

$d := 3$

$$3 \tag{4}$$

The following defines a function f as a function of variables x and y.

$f := (x, y) \rightarrow 4 \cdot x^3 + y^3 - 12 \cdot x - 3 \cdot y$

$$(x, y) \rightarrow 4\, x^3 + y^3 - 12\, x - 3\, y \tag{5}$$

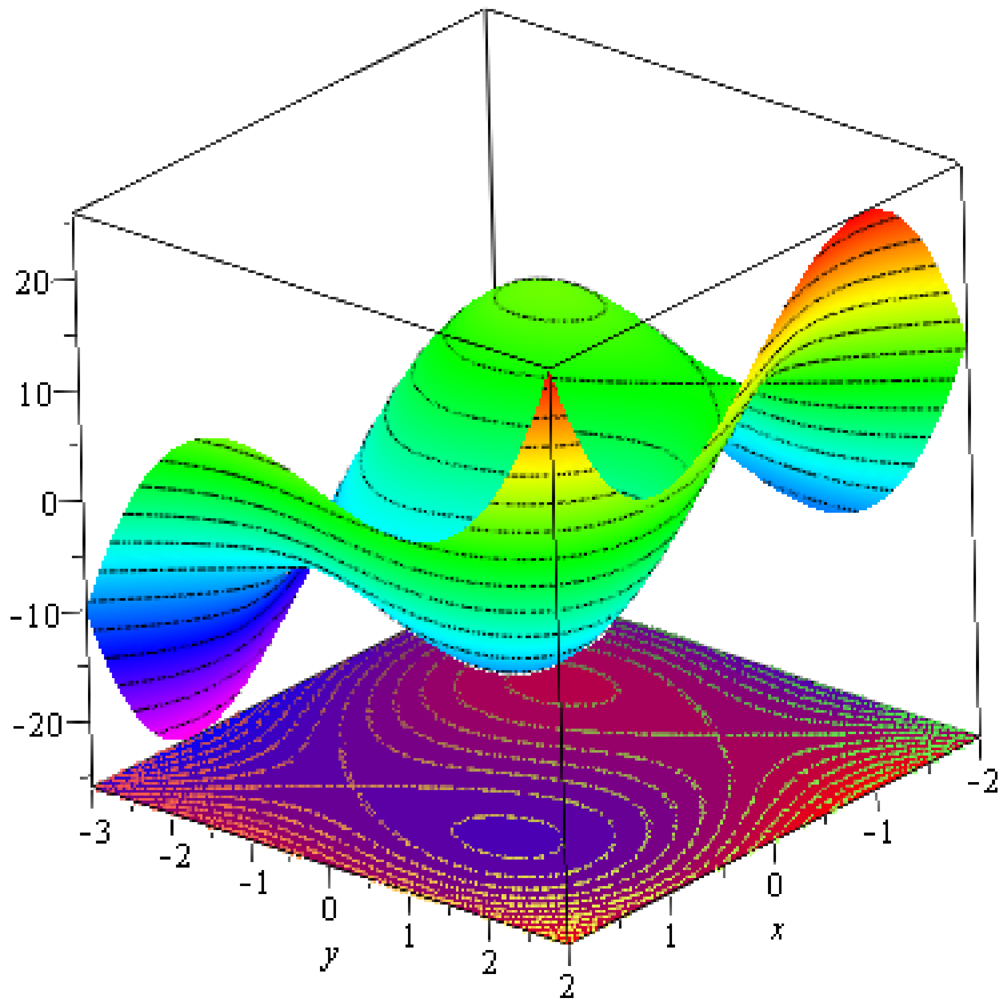
$p := plot3d(\, f(x, y), x = a\,..b, y = c\,..d, style = patchcontour, contours = [\,(3 \cdot k)\$k = -8\,..8\,], scaling = unconstrained, axes = boxed\,, transparency = 0.0, title = typeset(z = f(x, y)\,), titlefont = [\,TIMES, BOLD, 22\,], shading = zhue, grid = [\,100, 100\,]\,):$

$q := contourplot(\, f(x, y)\,, x = a\,..b, y = c\,..d, filled = true, thickness = 3, contours = [\,(3 \cdot k)\$k = -8\,..8\,], coloring = [\,blue, red\,], grid = [\,100, 100\,]\,):$

$F := transform(\ (x, y) \ \rightarrow\ [\ x, y,\ -26\,]\ ):$

$display(\ \{\ p, F(q)\ \}\ );$

$$z = 4x^3 + y^3 - 12x - 3y$$



Let's determine the critical points of function *f*. First determine $\frac{\partial}{\partial x} f$ and $\frac{\partial}{\partial y} f$.

$fx := D_1(f);$ # This means differentiate f wrt x and store it as function *fx(x,y)*.

$$(x, y) \to 12x^2 - 12 \quad \textbf{(6)}$$

$fxx := D_{1,1}(f);$ # This means differentiate *f* *wrt* *x* twice and store it as function *fxx(x,y)*.

$$(x, y) \to 24x \quad \textbf{(7)}$$

$fxy := D_{1,2}(f);$ # This means differentiate *f* *wrt* *x* and *y* and store it as function *fxx(x,y)*.

$$0 \quad \textbf{(8)}$$

$fy := D_2(f)$ # This means differentiate *f* *wrt* *y* and store it as function *fy(x,y)*.

$$(x, y) \to 3y^2 - 3 \quad \textbf{(9)}$$

$fyy := D_{2,2}(f)$ # This means differentiate f *wrt* y twice and store it as function $fyy(x,y)$.

$$(x, y) \to 6\,y \tag{10}$$

Let's determine d for the Second Derivative Test:

$d := fxx \cdot fyy - (fxy)^2;\ \ d(x, y)$

$$fxx\,fyy$$
$$144\,x\,y \tag{11}$$

Let's determine the critical points of function f.
We do so by setting $fx = 0$ and $fy = 0$ and solving simultaneously for x and y:

$solve(\{fx(x, y) = 0, fy(x, y) = 0\}, \{x, y\})$

$$\{x = 1, y = 1\}, \{x = 1, y = -1\}, \{x = -1, y = 1\}, \{x = -1, y = -1\} \tag{12}$$

So the critical points of f are at $(1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$.

Check the sign of d at the CP $(1, 1)$:
$d(1, 1)$

$$144 \tag{13}$$

Since $d(1, 1) > 0$, we must check the sign of $fxx(1, 1)$:
$fxx(1, 1)$

$$24 \tag{14}$$

Since d > 0 and $fxx > 0$ at $(1, 1)$, function f has a local minimum at point $CP(1, 1)$. That local minimum is:
$f(1, 1)$

$$-10 \tag{15}$$

Check the sign of d at the CP $(1, -1)$:
$d(1, -1)$

$$-144 \tag{16}$$

Since d < 0 at point $(1, -1)$, function f has a saddle point at $CP(1, -1)$, and there's nothing else to check.

Check the sign of d at the CP $(-1, 1)$:
$d(-1, 1)$

$$-144 \tag{17}$$

Since d < 0 at point $(-1, 1)$, function f has a saddle point at $CP(-1, 1)$, and there's nothing else to check.

Check the sign of d at the CP $(-1, -1)$:
$d(-1, -1)$

$$144 \tag{18}$$

Since $d(-1, -1) > 0$, we must check the sign of $fxx(-1, -1)$:
$fxx(-1, -1)$

$$-24 \tag{19}$$

Since d > 0 and $fxx < 0$ at $(-1, -1)$, function f has a local maximum at $CP(-1, -1)$. That local maximum is:
$f(-1, -1)$

$$10 \tag{20}$$