Consider the equation

$$x^2 + e^{y^2} + z^2 \;=\; \cos(xyz)\,, \tag{1}$$

which represents a surface in 3-D.

Determine $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Note that this implies that z is a function of variables x and y: $\boldsymbol{z = z(x, y)}$**.**

1. To determine $\dfrac{\partial z}{\partial x}$, differentiate both sides of Eq. (1) partially with respect to x, and then solve the resulting equation for $\dfrac{\partial z}{\partial x}$.

$$
\begin{aligned}
\frac{\partial}{\partial x}\Big(x^2 + e^{y^2} + z^2\Big) &= \frac{\partial}{\partial x}\Big[\cos(xyz)\Big] \\
\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}\Big(e^{y^2}\Big) + \frac{\partial}{\partial x}(z^2) &= -\sin(xyz)\,\frac{\partial}{\partial x}(xyz) \\
2x + 0 + 2z\frac{\partial z}{\partial x} &= -\sin(xyz)\,y\,\frac{\partial}{\partial x}(xz) && y \text{ acts as a constant} \\
2x + 2z\frac{\partial z}{\partial x} &= -\sin(xyz)\,y\,\Big(z + x\frac{\partial z}{\partial x}\Big) && \text{used product rule} \\
2x + 2zz_x &= -\sin(xyz)\,y\,(z + xz_x) && \text{switched to } z_x \text{ notation} \\
2x + 2zz_x &= -yz\sin(xyz) - xyz_x\sin(xyz) && \text{multiplied RHS} \\
2zz_x + xyz_x\sin(xyz) &= -2x - yz\sin(xyz) && \text{now solve for } z_x \\
z_x\Big[\,2z + xy\sin(xyz)\,\Big] &= -2x - yz\sin(xyz) && \text{factored } z_x \text{ on LHS} \\
z_x &= \frac{-2x - yz\sin(xyz)}{2z + xy\sin(xyz)}\,, && \text{so} \\
\frac{\partial z}{\partial x} &= -\,\frac{2x + yz\sin(xyz)}{2z + xy\sin(xyz)}\,.
\end{aligned}
$$

2. To determine $\dfrac{\partial z}{\partial y}$, differentiate both sides of Eq. (1) partially with respect to y, and then solve the resulting equation for $\dfrac{\partial z}{\partial y}$.

Remember that z is a <u>function</u> of variables x and y: $\quad z = z(x, y)$.

$$
\begin{aligned}
\frac{\partial}{\partial y}\Big(x^2 + e^{y^2} + z^2\Big) &= \frac{\partial}{\partial y}\Big[\cos(xyz)\Big] \\
\frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}\Big(e^{y^2}\Big) + \frac{\partial}{\partial y}(z^2) &= -\sin(xyz)\,\frac{\partial}{\partial y}(xyz) \\
0 + 2ye^{y^2} + 2z\frac{\partial z}{\partial y} &= -\sin(xyz)\,x\,\frac{\partial}{\partial y}(yz) && x \text{ acts as a constant} \\
2ye^{y^2} + 2z\frac{\partial z}{\partial y} &= -\sin(xyz)\,x\left(z + y\frac{\partial z}{\partial y}\right) && \text{used product rule} \\
2ye^{y^2} + 2zz_y &= -\sin(xyz)\,x\,(z + yz_y) && \text{switched to } z_y \text{ notation} \\
2ye^{y^2} + 2zz_y &= -xz\sin(xyz) - xyz_y\sin(xyz) && \text{multiplied RHS} \\
2zz_y + xyz_y\sin(xyz) &= -2ye^{y^2} - xz\sin(xyz) && \text{now solve for } z_y \\
z_y\Big[2z + xy\sin(xyz)\Big] &= -2ye^{y^2} - xz\sin(xyz) && \text{factored } z_y \text{ on LHS} \\
z_y &= \frac{-2ye^{y^2} - xz\sin(xyz)}{2z + xy\sin(xyz)}\,, && \text{so} \\
\frac{\partial z}{\partial y} &= -\,\frac{2ye^{y^2} + xz\sin(xyz)}{2z + xy\sin(xyz)}\,.
\end{aligned}
$$