The **partial derivative of f with respect to x** is defined by

$$\frac{\partial f}{\partial x} \equiv \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h},$$

provided the limit exists.

The **partial derivative of f with respect to y** is defined by

$$\frac{\partial f}{\partial y} \equiv \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h},$$

provided the limit exists.

EXAMPLE

Consider function $f(x,y) = 2x^3 + 3xy - y^2$.

1. Use the definition to determine the **partial derivative of f with respect to x**.
   First,

$$\begin{aligned} f(x+h,y) &= 2(x+h)^3 + 3(x+h)y - y^2 \\ &= 2(x^3 + 3x^2h + 3xh^2 + h^3) + 3(x+h)y - y^2 \\ &= 2x^3 + 6x^2h + 6xh^2 + 2h^3 + 3xy + 3hy - y^2, \end{aligned}$$

   and

$$\begin{aligned} f(x+h,y) - f(x,y) &= 2x^3 + 6x^2h + 6xh^2 + 2h^3 + 3xy + 3hy - y^2 - (2x^3 + 3xy - y^2) \\ &= 2x^3 + 6x^2h + 6xh^2 + 2h^3 + 3xy + 3hy - y^2 - 2x^3 - 3xy + y^2 \\ &= 6x^2h + 6xh^2 + 2h^3 + 3hy \\ &= h(6x^2 + 6xh + 2h^2 + 3y). \end{aligned}$$

   So

$$\begin{aligned} \frac{\partial f}{\partial x} &\equiv \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \\ &= \lim_{h \to 0} \frac{h(6x^2 + 6xh + 2h^2 + 3y)}{h} \\ &= \lim_{h \to 0} (6x^2 + 6xh + 2h^2 + 3y) \\ &= 6x^2 + 3y. \end{aligned}$$

2. Use the definition to determine the **partial derivative of f with respect to y**.

Recall that $f(x, y) = 2x^3 + 3xy - y^2$.

First,

$$\begin{aligned} f(x, y+h) &= 2x^3 + 3x(y+h) - (y+h)^2 \\ &= 2x^3 + 3x(y+h) - (y^2 + 2hy + h^2) \\ &= 2x^3 + 3xy + 3xh - y^2 - 2hy - h^2\,, \end{aligned}$$

and

$$\begin{aligned} f(x, y+h) - f(x, y) &= 2x^3 + 3xy + 3xh - y^2 - 2hy - h^2 - (2x^3 + 3xy - y^2) \\ &= 2x^3 + 3xy + 3xh - y^2 - 2hy - h^2 - 2x^3 - 3xy + y^2 \\ &= 3xh - 2hy - h^2 \\ &= h(3x - 2y - h)\,. \end{aligned}$$

So

$$\begin{aligned} \frac{\partial f}{\partial y} &\equiv \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h} \\ &= \lim_{h \to 0} \frac{h(3x - 2y - h)}{h} \\ &= \lim_{h \to 0} (3x - 2y - h) \\ &= 3x - 2y\,. \end{aligned}$$