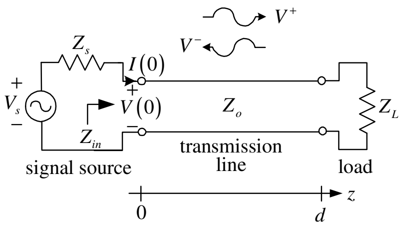
Helpful Summary of

Single Transmission Line, Sinusoidal Steady-State, Expressions

Introduction

Transmission lines have very interesting properties, quite different in many ways to those normally associated with standard circuits. These differences will be apparent in this concise overview of the important properties and equations used in the sinusoidal steady-state analysis of a single transmission line.



The transmission line in this discussion is to be considered two parallel conductors of length d, not too far apart, with uniform material around the conductors. A transmission line is also referred to as just a line or cable. The equations in this discussion can be used for twin-lead line, coaxial cable, some traces on printed circuit boards, and many other pairs of conductors. The passive load impedance, Z_L, at the output of the line, can be complex (i.e., contain both a real and an imaginary term):

$$Z_L = R_L + jX_L$$

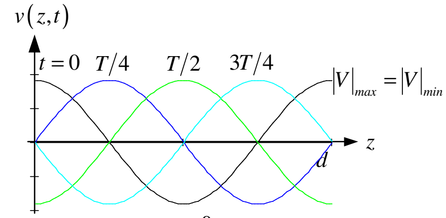
The source impedance, Z_s, can also be complex. The equations in this summary are for sinusoidal steady-state conditions. Thus, the input source, shown as V_s with a source impedance of Z_s, is in phasor form. In the real time domain this source voltage is

$$v_s(t) = A\cos(\omega t + \theta) = \mathrm{Re}\left[Ae^{j(\omega t+\theta)}\right] = \mathrm{Re}\left[V_s e^{j\omega t}\right]$$

where V_s is the phasor representation of the source voltage. This source voltage is not equal to the voltage at the input of the line at $z = 0$ unless $Z_s = 0$. The phasor V_s is equal to the amplitude A when θ is zero. It is important to recall from basic circuits that phasors are not a function of time. This simplifies the analysis but requires the use of the complex operator $j = \sqrt{-1}$. All of the voltage and current variables given in this discussion are phasors, including V^+ and $I(z)$, which is a phasor that is a function of z. Phasors have both an amplitude and a phase angle.

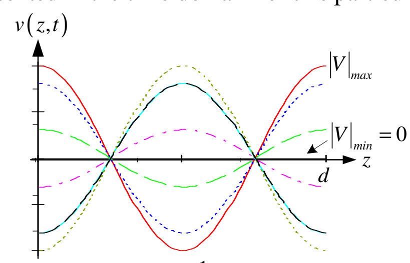
Traveling and Standing Waves

The voltage or current along a transmission line oriented parallel to the z axis can be described by the sum of (1) a wave traveling in the $+z$ direction such as $Ae^{-\alpha t}\cos(\omega t - \beta z + \theta_1)$, which is referred to as forward or positive-traveling wave, and (2) a wave traveling in the $-z$ direction such as $Be^{\alpha t}\cos(\omega t + \beta z + \theta_2)$, which is referred to as backward or negative-traveling wave. The figure that follows is a plot of a "pure" traveling wave: a wave that is only traveling in one direction, in this case the $+z$ direction with no attenuation or loss (i.e., $\alpha = 0$). What complicates the plotting of these traveling waves is that they are both a function of time, t, and position, z. Note that the maximum amplitude, $|V|_{max}$, and minimum amplitude, $|V|_{min}$, of the wave (its phasor amplitude) is the same. Obviously, the strength of the signal is varying with time, but its phasor amplitude (e.g., A) is constant.



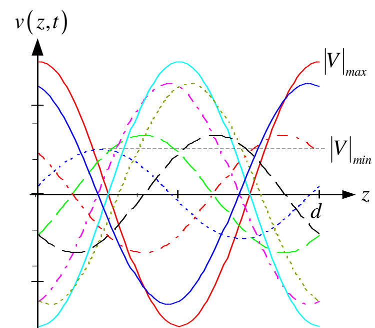
$\rho = 0$
pure traveling wave

When both forward and backward traveling waves are present on a line, a standing wave can be produced. The resultant wave stays in one location even as the time increases since the locations of the maximums and zeros appear to be fixed. Note that the total minimum phasor amplitude, $|V|_{min}$, is zero for a standing wave when represented in the time domain for this particular situation.



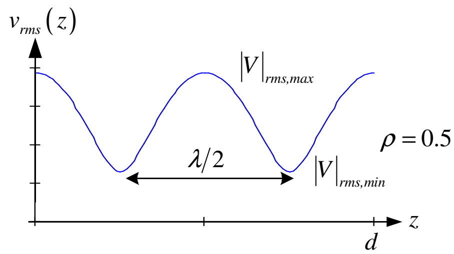
$\rho = 1$
standing wave

For other conditions, however, the two traveling waves do not appear to generate either a pure traveling or standing wave but a kind of combination of the two. An example of this situation follows. In this case, the minimum phasor amplitude, $|V|_{min}$, is not zero.



$\rho = 0.5$
standing and traveling wave

It is very convenient in practice to measure the rms value of a voltage. The rms value of a signal, which mathematically involves the time integration of the signal, is not a function of time. The rms value of a signal as a function of z is considered a standing wave even though it is not a function of time. This standing wave pattern is evident in the following plot.

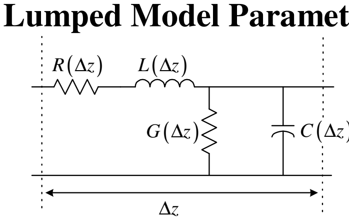


Variables and Units

variable	name and SI units
*	complex conjugate
–	negative-traveling wave indicator
$\angle$	phase angle of a complex number
+	positive-traveling wave indicator
A_{dB}	one-way attenuation in dB when $Z_o = Z_L$ (≥ 0)
C	capacitance per unit length (F/m)
d	distance to load (m)
G	conductance per unit length (S/m)
G_L	conductance of load when load is purely real
$I(z)$	phasor current in the upper conductor of the line at z (A)
Im	imaginary part
inc	incident to the load
j	complex operator $\sqrt{-1}$
L	inductance per unit length (H/m)
max	maximum
min	minimum
n	integers including zero $(0, \pm 1, \pm 2 \ldots)$
oc	open-circuited load ($Z_L = \infty$)
P_{avg}	average power (W)
Q	reactive power (VAR)
R	resistance per unit length (Ω/m)
Re	real part
ref	reflected from the load
R_L	resistance of load when load is purely real
rms	root mean squared
S	complex power (VA)
s	voltage standing wave ratio (VSWR)
sc	short-circuited load ($Z_L = 0$)
$\tan^{-1}$	inverse tangent or arctangent
v	velocity (m/s)
$V(0)$	phasor voltage across the input of the line also referred to as the sending-end voltage (V)
$V(d)$	phasor voltage across the output of the line also referred to as the receiving-end voltage (V)
$V(z)$	phasor voltage across the line at z (V)
V^+	incident or forward component of traveling voltage phasor (V)
V^-	reflected or backward component of traveling current phasor (V)
z	location along the line (m)
Z_{in}	input impedance (Ω)
Z_{in}^L	input impedance with the load Z_L (Ω)

z_{min}	location of first input impedance minimum from load (m)
Z_L	load impedance (Ω)
Z_o	characteristic impedance (Ω)
$Z_o \neq Z_L$	mismatched
$Z_o = Z_L$	matched
Z_s	source impedance (Ω)
α	attenuation constant (nepers/m)
β	phase constant (rad/m)
γ	propagation constant = $\alpha + j\beta$ (1/m)
T	transmission coefficient
λ	wavelength (m)
ρ	(voltage) reflection coefficient
ϕ	phase angle of reflection coefficient (rad)
ω	radian frequency (rad/s)

Lumped Model Parameters



An electrically small segment of a line of length Δz $(<< \lambda/10)$ is shown. The parameters R, L, G, and C are per unit length. The resistance of both conductors is modeled by R. The resistance of the dielectric between the two conductors is modeled by the conductance, G, which is *not* equal to $1/R$. (As G decreases, the dielectric losses decrease.) The capacitance between the conductors is modeled by C, and the inductance of the path generated by the two conductors is modeled by L. A line is considered lossless when both R and G are zero (or negligible). A transmission line is not a single lumped circuit as shown but the sum of many of these lumped circuits. (The number of these segments for a line of length d would be $d/\Delta z$.) The resultant representation for the line as Δz approaches zero is referred to as a distributed circuit.

Characteristic Impedance

The characteristic impedance of a line, Z_o, is probably the most important parameter of a line. This is the value often printed on common coaxial cable (e.g., 50 Ω or 75 Ω). The characteristic impedance is <u>not</u> the total resistance or impedance of the cable, although it is often referred to as the line impedance, and it is <u>not</u> a function of the length of the cable.

$$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{\left(RG + \omega^2 LC\right) + j\omega\left(LG - RC\right)}{G^2 + \left(\omega C\right)^2}}$$

Generally, Z_o is a complex value, but at higher frequencies it is approximately $\sqrt{L/C}$. This real expression is the most commonly used equation for the characteristic impedance. The following expressions are useful since they do not involve the square root of a complex number or function.

$$Z_o \approx \begin{cases} \sqrt{R/G} \text{ if } R >> \omega L, G >> \omega C \\ \sqrt{R/(2\omega C)}(1-j) \text{ if } R >> \omega L, G << \omega C \\ \sqrt{\omega L/(2G)}(1+j) \text{ if } R << \omega L, G >> \omega C \\ \sqrt{L/C} \text{ if } R << \omega L, G << \omega C \end{cases}$$

Reflection Coefficient

The reflection coefficient is a measure of the closeness of Z_L to Z_o. It is generally also a complex quantity unless both the load is purely resistive and the line is lossless. When the load is equal to the line impedance, the line is referred to as matched and the reflection coefficient is zero. A wave on a line impinging on a matched load will not reflect off the load.

$$\rho = \frac{Z_L - Z_o}{Z_L + Z_o} = |\rho|\angle\phi = \begin{cases} +1 \text{ if } Z_L = \infty \\ -1 \text{ if } Z_L = 0 \\ 0 \text{ if } Z_L = Z_o \\ |\rho| = 1 \text{ if line lossless \&} \\ \text{ load purely reactive} \end{cases}$$

$$Z_L = Z_o \frac{1+\rho}{1-\rho}$$

At the load the reflection coefficient is the ratio of the reflected voltage component, V^-, to the incident voltage component, V^+. (A reflection coefficient can also be defined for the current.) Although not used in this summary, the transmission coefficient is defined as $1 + \rho$.

Propagation Constant

The propagation constant determines how quickly a wave attenuates or decays over the length of the line. It also determines how the phase of the wave changes over the length of the line. The attenuation or loss is determined by α and the phase change by β.

$$\gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)}$$
$$= \sqrt{(RG - \omega^2 LC) + j\omega(RC + LG)}$$
$$= \frac{1}{d}\tanh^{-1}\sqrt{\frac{Z_{in}^{sc}}{Z_{in}^{oc}}} = \frac{R + j\omega L}{Z_o} = Z_o(G + j\omega C)$$

The following approximations are useful since they do not involve the square root of a complex number or function.

$$\gamma \approx \begin{cases} \sqrt{RG} \text{ if } R >> \omega L, G >> \omega C \\ \sqrt{\omega RC/2}(1+j) \text{ if } R >> \omega L, G << \omega C \\ \sqrt{\omega GL/2}(1+j) \text{ if } R << \omega L, G >> \omega C \\ j\omega\sqrt{LC} \text{ if } R << \omega L, G << \omega C \end{cases}$$

Lossless Line Basics

If the losses of a line are small, which is often the case if the frequency is not too small or too high, then the following relationships are used. Note that the velocity of a wave on the line, v, and the characteristic impedance, Z_o, are not a function of the frequency.

$$\alpha = 0, \ \ \beta = 2\pi/\lambda = \omega\sqrt{LC},$$
$$v = 1/\sqrt{LC} = \omega/\beta, \ \ Z_o = \sqrt{L/C}, \ \ |\rho| \le 1$$

Input Impedance (Lossless Lines)

The impedance looking into a transmission line is not necessarily equal to Z_o or Z_L. The impedance looking toward the load along the line is periodic with a period of $\lambda/2$. When the load is either a short or an open circuit, then the input impedance is either inductive or capacitive with no real component since there are no resistive “sources” along the lossless line. When the load is resistive or complex, then the input impedance can be “nearly” anything. Its value is a function of the frequency, line impedance, load impedance, phase constant, and length. For the often desirable matched condition, $Z_L = Z_o$, the input impedance is always Z_o. When $d = \lambda/4$, the transmission line is referred to as a quarter-wave transformer.

$$Z_{in} = \frac{V(0)}{I(0)} = Z_o \frac{Z_L + jZ_o\tan(\beta d)}{Z_o + jZ_L\tan(\beta d)} = Z_o \frac{1+\rho e^{-j2\beta d}}{1-\rho e^{-j2\beta d}}$$

$$Z_{in} = \begin{cases} -jZ_o/\tan(\beta d) \text{ if } Z_L = \infty \\ jZ_o\tan(\beta d) \text{ if } Z_L = 0 \\ Z_o \text{ if } Z_L = Z_o \\ Z_o(Z_L + jZ_o)/(Z_o + jZ_L) \text{ if } d = \lambda/8 \Rightarrow |Z_{in}| = |Z_o| \\ Z_o^2/Z_L \text{ if } d = \lambda/4 \ \ \Rightarrow Z_o = \sqrt{Z_{in}Z_L} \\ Z_L \text{ if } d = \lambda/2 \\ Z_L \text{ if } d = \lambda \end{cases}$$

When the line is electrically short and the load is purely resistive, the line impedance is resistive and inductive or resistive and capacitive dependent on the size of the load:

$$Z_{in} = \begin{cases} R_L + j\omega dL\left[1 - (R_L^2 C/L)\right] \text{ if } \beta d << 1, Z_o >> R_L \\ \left\{G_L + j\omega dC\left[1 - (G_L^2 L/C)\right]\right\}^{-1} \text{ if } \beta d << 1, Z_o << R_L \end{cases}$$

Interestingly, the maximum amplitude of the impedance looking toward the load is at the voltage maximums along the line. The minimum amplitude of this impedance is at the voltage minimums along the line. Furthermore, these maximum and minimum values are entirely real at these locations. In order to obtain these extreme values, the line must be sufficiently long so that these values are reached.

$$Z_{in,max} = \frac{|V|_{max}}{|I|_{min}} = sZ_o, \ \ Z_{in,min} = \frac{|V|_{min}}{|I|_{max}} = \frac{Z_o}{s}$$

$$Z_L = Z_o \frac{Z_{in} - jZ_o\tan(\beta d)}{Z_o - jZ_{in}\tan(\beta d)} = Z_{in}^{oc}\frac{Z_{in} - Z_{in}^{sc}}{Z_{in}^{oc} - Z_{in}}$$

$$Z_L = Z_o \frac{1 - js\tan(\beta z_{min})}{s - j\tan(\beta z_{min})}$$

The characteristic impedance can also be obtained by measuring the input impedance of the line when the load is a short circuit and an open circuit.

$$Z_o = \sqrt{Z_{in}^{sc}Z_{in}^{oc}}$$

Voltage (Lossless Lines)

The voltage across the line is described by a differential equation referred to as the transmission line equation:

$$\frac{dV^2(z)}{dz^2} = -\beta^2 V(z)$$

When this differential equation is solved, it is determined that the total voltage across the conductors is given by the sum of a positive-traveling wave, $V^+e^{-j\beta z}$, and a negative-traveling wave, $V^-e^{j\beta z}$. Although the voltage waveform is spatially periodic, repeating every wavelength, λ, its amplitude repeats every $\lambda/2$.

$$V(z) = V^+e^{-j\beta z} + V^-e^{j\beta z} = V^+e^{-j\beta z}\left[1+\rho(z)\right]$$
$$= V^+e^{-j\beta z}\left[1+\rho e^{j2\beta(z-d)}\right] \text{ where } V^- = V^+\rho e^{-j2\beta d}$$

$$|V(z)| = \begin{cases} |V^+|\sqrt{1+|\rho|^2+2|\rho|\cos\left[2\beta(z-d)+\phi\right]} \\ |V(z+(\lambda/2))| \\ 2\left|V^+\cos\left[\beta(z-d)\right]\right| \text{ if } Z_L=\infty \\ 2\left|V^+\sin\left[\beta(z-d)\right]\right| \text{ if } Z_L=0 \\ |V^+| \text{ if } Z_L = Z_o \end{cases}$$

$$V^+ = \frac{1}{2}\left[V(0)+Z_oI(0)\right] = \frac{1}{2}\left[V(d)+Z_oI(d)\right]e^{j\beta d}$$
$$V^- = \frac{1}{2}\left[V(0)-Z_oI(0)\right] = \frac{1}{2}\left[V(d)-Z_oI(d)\right]e^{-j\beta d}$$

The phase angle at $\lambda/4$ multiples along the line relative to the phase angle at the load is

$$\angle V(d)-\angle V(d-(\lambda/4)) = \tan^{-1}(X_L/R_L)-90^\circ$$
$$\angle V(d)-\angle V(d-(\lambda/2)) = -180^\circ$$
$$\angle V(d)-\angle V(d-(3\lambda/4)) = \tan^{-1}(X_L/R_L)+90^\circ$$
$$\text{where } \angle Z_L = \tan^{-1}(X_L/R_L) = \angle V(d)-\angle I(d)$$

The maximum amplitude of the voltage occurs at the load when $Z_L > Z_o$ and Z_L is purely resistive:

$$|V|_{max} = |V^+|(1+|\rho|) = \left|\frac{V(d)(Z_L+Z_o)}{2Z_L}\right|(1+|\rho|)$$
$$\text{occurs at } z = d+\left[(2n\pi-\phi)/(2\beta)\right]$$
$$\text{occurs at } z = d \text{ if } Z_L > Z_o \,\&\, Z_L \text{ real}$$

The minimum amplitude of the voltage occurs at the load when $Z_L < Z_o$ and Z_L is purely resistive:

$$|V|_{min} = |V^+|(1-|\rho|) = \left|\frac{V(d)(Z_L+Z_o)}{2Z_L}\right|(1-|\rho|)$$
$$\text{occurs at } z = d+\left\{\left[(2n-1)\pi-\phi\right]/(2\beta)\right\}$$
$$\text{occurs at } z = d \text{ if } Z_L < Z_o \,\&\, Z_L \text{ real}$$

When the load and line impedance are equal, the maximum and minimum amplitudes are equal:

$$|V|_{max} = |V|_{min} \Rightarrow \text{matched}$$

Current (Lossless Lines)

The current along the line is also described by the transmission line equation:

$$\frac{dI^2(z)}{dz^2} = -\beta^2 I(z)$$

The solution to this differential equation for the current is similar to the voltage expression. It is not, however, merely equal to $V(z)/Z_o$ unless the line is matched. Note the – sign in front of the negative-traveling term. The amplitude of the current, as with the voltage, has a spatial period of $\lambda/2$.

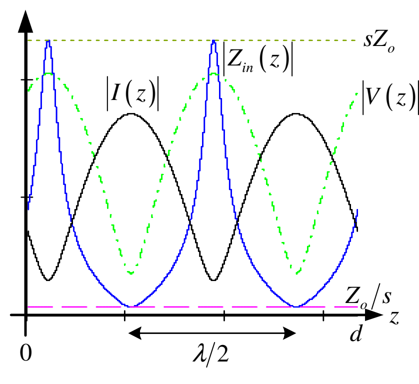
$$I(z) = \frac{V^+e^{-j\beta z}}{Z_o} - \frac{V^-e^{j\beta z}}{Z_o} = \frac{V^+e^{-j\beta z}\left[1-\rho(z)\right]}{Z_o}$$
$$= \frac{V^+e^{-j\beta z}\left[1-\rho e^{j2\beta(z-d)}\right]}{Z_o}$$

$$|I(z)| = \begin{cases} \left(|V^+|/Z_o\right)\sqrt{1+|\rho|^2-2|\rho|\cos\left[2\beta(z-d)+\phi\right]} \\ |I(z+(\lambda/2))| \\ (2/Z_o)\left|V^+\sin\left[\beta(z-d)\right]\right| \text{ if } Z_L=\infty \\ (2/Z_o)\left|V^+\cos\left[\beta(z-d)\right]\right| \text{ if } Z_L=0 \\ |V^+|/Z_o \text{ if } Z_L = Z_o \end{cases}$$

$$\angle I(d)-\angle I(d-(\lambda/4)) = -90^\circ - \tan^{-1}(X_L/R_L)$$
$$\angle I(d)-\angle I(d-(\lambda/2)) = -180^\circ$$
$$\angle I(d)-\angle I(d-(3\lambda/4)) = 90^\circ - \tan^{-1}(X_L/R_L)$$

$$|I|_{max} = \frac{|V^+|}{Z_o}(1+|\rho|) = \frac{|V|_{max}}{Z_o}$$
$$|I|_{min} = \frac{|V^+|}{Z_o}(1-|\rho|) = \frac{|V|_{min}}{Z_o},$$
$$Z_o = \frac{|V|_{max}}{|I|_{max}} = \frac{|V|_{min}}{|I|_{min}}$$

The current amplitude minimum is located at the voltage amplitude maximum, and the current maximum at the voltage minimum. The relationship between the current, voltage, and input impedance is shown in the given figure for a mismatched line (the variables are scaled by different factors).



Power (Lossless Lines)

The degree of mismatch between the load and line determines the power that is actually delivered to the load. Interestingly, the time-average power along a lossless line is not a function of z. This is reasonable since there are no line losses. However, the reactive power, Q, is a function of z. The sign of this imaginary power is an indication of whether the stored energy is inductive (+) or capacitive (–) for the given location, z. The power absorbed by the load is given as $P_{L,avg}$.

$$S = \frac{1}{2}V(z)I^*(z) = P_{avg} + jQ$$

$$P_{avg} = \frac{1}{2}\mathrm{Re}\left[V(z)I^*(z)\right] = \mathrm{Re}\left[V_{rms}(z)I_{rms}^*(z)\right]$$

$$P_{avg} = \frac{\left|V^+\right|^2}{2Z_o}\left(1-\left|\rho\right|^2\right) = P_{avg}^+ + P_{avg}^- = \frac{\left|V^+\right|^2}{2Z_o} - \frac{\left|V^-\right|^2}{2Z_o}$$

$$Q = \frac{1}{2}\mathrm{Im}\left[V(z)I^*(z)\right] = \mathrm{Im}\left[V(z)I^*(z)\right]$$

$$= \frac{\left|V^+\right|^2}{Z_o}\left|\rho\right|\sin\left[2\beta(z-d)+\phi\right]$$

$$P_{L,avg} = \frac{\left|V^+\right|^2}{2Z_o}\left(1-\left|\rho\right|^2\right) = \frac{\left|V^+\right|^2 - \left|V^-\right|^2}{2Z_o}$$

$$P_{inc,avg} = \frac{\left|V^+\right|^2}{2Z_o}, \quad P_{ref,avg} = \frac{\left|V^-\right|^2}{2Z_o}$$

$$\frac{P_{L,avg}}{P_{inc,avg}} = \frac{P_{inc,avg} - P_{ref,avg}}{P_{inc,avg}} = \frac{\left|V^+\right|^2 - \left|V^-\right|^2}{\left|V^+\right|^2} = 1 - \left|\rho\right|^2 = \frac{4s}{(s+1)^2}$$

$$\sqrt{\frac{P_{ref,avg}}{P_{inc,avg}}} = \frac{\left|V^-\right|}{\left|V^+\right|} = \frac{\left|V_{ref}\right|}{\left|V_{inc}\right|} = \left|\rho\right|$$

The incident power is that power contained in the wave traveling toward the load, while the reflected power is that power contained in the wave reflected off the source in the direction of the input of the line.

$$return\ loss_{dB} = -10\log\left(\frac{P_{ref,avg}}{P_{inc,avg}}\right) = -10\log\left|\rho\right|^2$$

$$= -20\log\left|\rho\right| = -20\log\left(\frac{s-1}{s+1}\right)$$

$$reflected\ loss_{dB} = -10\log\left(\frac{P_{L,avg}}{P_{inc,avg}}\right) = -10\log\left(1-\left|\rho\right|^2\right)$$

Standing Wave Ratio (Lossless Lines)

The standing wave ratio is often measured to determine the relative degree of mismatch between the load and line. It can be defined in terms of the ratio of the voltage magnitude extremes or current magnitude extremes. The standing wave ratio is always greater than or equal to one. When the line is matched, the standing wave ratio is equal to one, the best that it can be. Generally, it is desirable to have a low standing wave ratio.

$$s = \frac{\left|V\right|_{max}}{\left|V\right|_{min}} = \frac{\left|I\right|_{max}}{\left|I\right|_{min}} = \frac{1+\left|\rho\right|}{1-\left|\rho\right|} = \frac{1+\left|\dfrac{Z_L - Z_o}{Z_L + Z_o}\right|}{1-\left|\dfrac{Z_L - Z_o}{Z_L + Z_o}\right|}$$

$$s = \begin{cases} \infty \ \text{ if } Z_L = \infty \\ \infty \ \text{ if } Z_L = 0 \\ 1 \ \text{ if } Z_L = Z_o \\ Z_L/Z_o \ \text{ if } Z_L > Z_o \text{ \& load purely real} \\ Z_o/Z_L \ \text{ if } Z_o > Z_L \text{ \& load purely real} \end{cases}$$

$$\rho = \frac{s-1}{s+1}e^{j[2\beta z_{min} \pm \pi]}$$

$$\left|\rho\right| = \frac{s-1}{s+1} = \frac{\left|V\right|_{max} - \left|V\right|_{min}}{\left|V\right|_{max} + \left|V\right|_{min}}$$

Lossy Line Basics

For a lossy line where $\alpha \neq 0$, the waves do not exactly repeat with z since there is attenuation along the line. The phase velocity is in general a function of frequency, $v = \omega/\beta$. When the velocity is a function of frequency, distortion will occur for signals with more than one frequency component (e.g., a square wave or exponential pulse). For the special case where $L/R = C/G$, the velocity is independent of the frequency and Z_o is real. This is referred to as a distortionless line, even though there are losses on the line. For a lossy line the reflection coefficient can be greater than one (but no greater than 2.41).

Voltage (Lossy Lines)

The voltage across a lossy line is described by a differential equation that is similar to the lossless case:

$$\frac{dV^2(z)}{dz^2} = \gamma^2 V(z)$$

The voltage across the line can be written in terms of the hyperbolic sine and cosine functions:

$$V(z) = V^+e^{-\gamma z} + V^-e^{\gamma z} = V^+e^{-\gamma z}\left[1+\rho(z)\right]$$

$$= V^+e^{-\gamma z}\left[1+\rho e^{2\gamma(z-d)}\right]$$

$$= V(0)\cosh(\gamma z) - I(0)Z_o\sinh(\gamma z)$$

$$= V(d)\cosh\left[\gamma(z-d)\right] - I(d)Z_o\sinh\left[\gamma(z-d)\right]$$

When the line is matched to its load, there is no reflection, only a damped forward-traveling wave. For the three special cases that follow, the source impedance is equal to the line impedance:

$$\left|V(z)\right| = \begin{cases} \left|V_s\right|e^{-\alpha d}\left|\cosh\left[\gamma(z-d)\right]\right| \ \text{ if } Z_L = \infty, Z_s = Z_o \\ \left|V_s\right|e^{-\alpha d}\left|\sinh\left[\gamma(z-d)\right]\right| \ \text{ if } Z_L = 0, Z_s = Z_o \\ \left(\left|V_s\right|/2\right)e^{-\alpha z} \ \text{ if } Z_L = Z_o, Z_s = Z_o \end{cases}$$

$$V^+ = \frac{1}{2}\left[V(0) + Z_oI(0)\right] = \frac{1}{2}\left[V(d) + Z_oI(d)\right]e^{\gamma d}$$

$$V^- = \frac{1}{2}\left[V(0) - Z_oI(0)\right] = \frac{1}{2}\left[V(d) - Z_oI(d)\right]e^{-\gamma d}$$

Current (Lossy Lines)

$$\frac{dI^2(z)}{dz^2} = \gamma^2 I(z)$$

The current along the line can be written in terms of the hyperbolic sine and cosine functions:

$$I(z) = \frac{V^+}{Z_o}e^{-\gamma z} - \frac{V^-}{Z_o}e^{\gamma z} = \frac{V^+}{Z_o}e^{-\gamma z}\left[1-\rho(z)\right]$$

$$= \frac{V^+}{Z_o}e^{-\gamma z}\left[1-\rho e^{2\gamma(z-d)}\right]$$

$$= I(0)\cosh(\gamma z) - \frac{V(0)}{Z_o}\sinh(\gamma z)$$

$$= I(d)\cosh\left[\gamma(z-d)\right] - \frac{V(d)}{Z_o}\sinh\left[\gamma(z-d)\right]$$

When the line is matched to its load, there is no reflection, only a damped forward-traveling wave. For the three special cases that follow, the source impedance is equal to the line impedance:

$$|I(z)| = \begin{cases} |V_s/Z_o| e^{-\alpha d} \left|\sinh\left[\gamma(z-d)\right]\right| & \text{if } Z_L = \infty,\ Z_s = Z_o \\ |V_s/Z_o| e^{-\alpha d} \left|\cosh\left[\gamma(z-d)\right]\right| & \text{if } Z_L = 0,\ Z_s = Z_o \\ |V_s/(2Z_o)| e^{-\alpha z} & \text{if } Z_L = Z_o,\ Z_s = Z_o \end{cases}$$

Input Impedance (Lossy Lines)

$$Z_{in} = Z_o \frac{Z_L + Z_o \tanh(\gamma d)}{Z_o + Z_L \tanh(\gamma d)} = Z_o \frac{1+\rho e^{-2\gamma d}}{1-\rho e^{-2\gamma d}}$$

$$= \begin{cases} Z_o / \tanh(\gamma d) & \text{if } Z_L = \infty \\ Z_o \tanh(\gamma d) & \text{if } Z_L = 0 \\ Z_o & \text{if } Z_L = Z_o \end{cases}$$

The parameters Z_o, α, and β of a lossy line can be determined by measuring the input impedance for both a short-circuited and an open-circuited load. Instead of using an open-circuited load to determine Z_o, an arbitrary Z_L can be used. An "infinite" load impedance is not that simple to obtain at higher frequencies due to the load having nonzero capacitance.

$$Z_L = Z_o \frac{Z_{in} - Z_o \tanh(\gamma d)}{Z_o - Z_{in} \tanh(\gamma d)} = Z_{in}^{oc} \frac{Z_{in} - Z_{in}^{sc}}{Z_{in}^{oc} - Z_{in}}$$

$$Z_o = \sqrt{\frac{Z_L Z_{in}^{sc} Z_{in}^{L}}{Z_L + Z_{in}^{sc} - Z_{in}^{L}}} = \sqrt{Z_{in}^{sc} Z_{in}^{oc}}$$

$$\alpha = \frac{1}{2d} \ln \left| \frac{1+\sqrt{Z_{in}^{sc}/Z_{in}^{oc}}}{1-\sqrt{Z_{in}^{sc}/Z_{in}^{oc}}} \right|$$

$$\beta = \frac{1}{2d}\left[\angle\left(\frac{1+\sqrt{Z_{in}^{sc}/Z_{in}^{oc}}}{1-\sqrt{Z_{in}^{sc}/Z_{in}^{oc}}}\right) + 2n\pi\right]$$

Power (Lossy Lines)

In general, the forward and reflected powers cannot be merely added as with the lossless line: $P_{avg} \neq P_{avg}^{+} + P_{avg}^{-}$

$$S = \frac{1}{2} V(z) I^*(z) = P_{avg} + jQ$$

$$S^+ = \frac{1}{2} V^+(z)\left[I^+(z)\right]^* = \frac{|V^+|^2}{2Z_o^*} e^{-2\alpha z}$$

$$A_{dB} = 10 \log\left(e^{2\alpha d}\right) \approx 8.68\alpha d$$

Standing Wave Ratio (Lossy Lines)

Generally, there is not a single value for the VSWR on a lossy line since the values of the maximum and minimum amplitudes are varying along the line. The maximum and minimum *possible* values of these amplitudes (i.e., the envelopes) are

$$|V(z)|_{max,\,possible} = |V^+| e^{-\alpha z} \left|1+|\rho| e^{2\alpha(z-d)}\right|$$

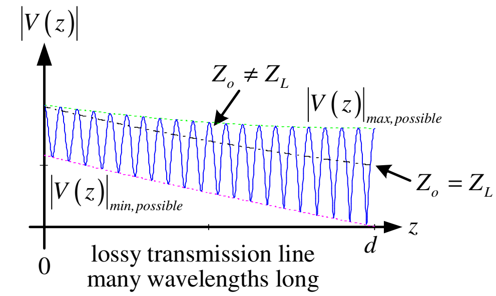
$$|V(z)|_{min,\,possible} = |V^+| e^{-\alpha z} \left|1-|\rho| e^{2\alpha(z-d)}\right|$$

However, if the line is sufficiently long so that there are many spatial waves of the voltage/current, then an adjacent maximum and minimum can be used to determine a reasonable local VSWR.

$$s = \frac{|V(z)|_{max}}{|V(z)|_{min}} \triangleq \frac{1+|\rho| e^{2\alpha(z-d)}}{1-|\rho| e^{2\alpha(z-d)}}$$

$$= \frac{1+|\rho| 10^{-A_{dB}/10}}{1-|\rho| 10^{-A_{dB}/10}} \quad \text{if } z = 0$$

The one-way loss or attenuation along a lossy line in dB, A_{dB}, is often given per unit length (e.g., 1.6 dB/100 m) assuming a matched load. The total loss in dB for a matched load is then this loss multiplied by the length of the line. When the load is not matched to the line, the loss is greater than this matched-loss value: as the standing wave ratio increases, the losses increase. The difference between the maximum and minimum voltage magnitudes (and currents) decreases when moving away from the load. Even when the line is matched to the load, the magnitude of the voltage varies along the length of the line, being largest at the source or input.



Although the VSWR might be mostly conveniently measured at the input of the line, the VSWR at the load can be significantly greater for a lossy line.

Auxiliary Relationships

$$|A + jB| = \left|\sqrt{A^2+B^2} \angle \tan^{-1}(B/A)\right| = \sqrt{A^2+B^2}$$

$$\sqrt{A + jB} = \sqrt{\sqrt{A^2+B^2} \angle \tan^{-1}(B/A)}$$

$$= \sqrt{\sqrt{A^2+B^2}} \angle (1/2) \tan^{-1}(B/A)$$

$$(A + jB)^* = A - jB = \sqrt{A^2+B^2} \angle -\tan^{-1}(B/A)$$

$$s(t) = \mathrm{Re}\left[S e^{j\omega t}\right] = \mathrm{Re}\left[(A + jB) e^{j\omega t}\right]$$

$$= \sqrt{A^2+B^2} \cos\left[\omega t + \tan^{-1}(B/A)\right]$$

$$s(t,z) = \mathrm{Re}\left[A e^{-(\alpha + j\beta) z} e^{j\omega t}\right] = A e^{-\alpha z} \cos(\omega t - \beta z)$$

$A e^{\mp \alpha z} \cos(\omega t \mp \beta z)$ decaying wave traveling in the $\pm z$ direction

$X_{rms} = X/\sqrt{2}$ valid for sinusoidal signals with zero dc offset

$R \Leftrightarrow R = Z_R$, $L \Rightarrow j\omega L = Z_L$, $C \Leftrightarrow 1/j\omega C = Z_C$

$V_{dB} = 20 \log V$, $P_{dB} = 10 \log P$

$\omega = 2\pi f$, $\lambda = v/f$, $e^{jx} = \cos x + j \sin x$, $j = \sqrt{-1}$

$\cosh(x) = \left(e^x + e^{-x}\right)/2$, $\sinh(x) = \left(e^x - e^{-x}\right)/2$

$\tanh(x) = \left(e^x - e^{-x}\right)/\left(e^x + e^{-x}\right)$

0.3048 m = 1 ft, 1 mile = 5,280 ft, $c \approx 3 \times 10^8$ m/s