Chapter 12: Spectra of Periodic and Aperiodic Signals

12.1 Determine whether each of the following functions are periodic. If they are periodic, provide their fundamental frequency and period.

a) $x(t) = 4\cos(5t) - 2\sin(10t)$

b) $p(t) = 4\cos(5t) - 2\sin(10t) - 7$

b) $k(t) = 3\cos(\pi t) + 6\sin(10\pi t) - 7\cos(15\pi t + 45^\circ)$

c) $j(t) = 4\cos\left(5\sqrt{2}t\right) - 2\sin\left(10t + \dfrac{\pi}{5}\right)$

d) $n(t) = 3\sin^2(30\pi t) + 6\sin(30\pi t) - 7\cos(35\pi t + 15^\circ)\cos(20\pi t + 185^\circ)$

e) $m(t) = 3\cos^3(30\pi t) + -7\cos(60\pi t + 105^\circ)\sin(15\pi t - 85^\circ) + 12$

f) $w(t) = 5e^{j10t} + e^{-3\cos(10t)}$

g) $p(t) = 3\sin^2(4t) + 2\cos(2t)e^{j10t}$

12.2 Determine the fundamental frequency in Hz of the waveform represented by the following Fourier series.

$$\sum_{n=-\infty}^{\infty} \frac{1 - e^{-6\left(\frac{\alpha + j2\pi n}{3\pi}\right)}}{4 + j2\pi n} e^{j\frac{4n}{3}t}$$

12.3 Determine the (one- or single-sided) amplitude of the 3rd harmonic of the waveform represented by the following Fourier series.

$$\frac{3A}{4} - \sum_{n=1}^{\infty} \left\{ \frac{2}{(2n-1)^2} \cos\left[\frac{2\pi(2n-1)}{T} t\right] + \frac{\pi}{n} \sin\left(\frac{2\pi n}{T} t\right) \right\}$$

12.4C For the periodic function, with at least one discontinuity per period, given in the Fourier series table in this chapter (the specific function provided by the instructor), plot N = 5, 10, 100, and 500 terms of the series. Verify that the maximum overshoot is about 9% of the height of each discontinuity for each N.

12.5C For the periodic function given in the Fourier series table in this chapter (the specific function provided by the instructor), plot the percent energy contained within the first N terms of the series versus N for N = 1 to 500 in 1 term increments. Then, determine the value of N corresponding to about 90% percent energy (over one period).

12.6C For the periodic function given in the Fourier series table in this chapter (the specific function provided by the instructor), verify via the integral relationships, all forms of the series provided in the table. Do not use any symmetry arguments to reduce the number or complexity of the integrations. Also, if specific dc, fundamental, and harmonic terms are given, verify that they are correct. Finally, if the function has any symmetry, verify that *all* of properties (e.g., even harmonics are zero) related to the symmetry condition are satisfied.

12.7 For the periodic function given in the Fourier series table in this chapter (the specific function provided by the instructor), check the expression against another series given in the table by using the dc shift, linearity, time reversal, time shifting, time differentiation, time integration, function multiplication properties or any combination of these properties. Do not use the definition for the Fourier series as a check.

12.8 Using the conversion relationships, verify each of the complex coefficients in Table 1 using the trigonometric coefficients. Then, verify each of the trigonometric coefficients using the complex coefficients.

Table 1

n	a_n	b_n	F_n	F_{-n}
0	1.25	-	0.624	-
1	0	0.975	$-0.488j$	$0.488j$
2	−0.401	0	−0.2	−0.2
3	0	0.022	$-0.011j$	$0.011j$
4	−0.104	0	−0.052	−0.052
5	0	−0.017	$0.0084j$	$-0.0084j$

12.9 Using the conversion relationships, verify each of the complex coefficients in Table 2 using the trigonometric coefficients. Then, verify each of the trigonometric coefficients using the complex coefficients.

Table 2

n	a_n	b_n	F_n	F_{-n}
0	0.00211	-	0.00106	-
1	1.27	0	0.637	0.637
2	−0.00217	0.00173	$-0.00109 - j0.000865$	$-0.00109 + j0.000865$
3	−0.424	0	−0.212	−0.212
4	0.00235	−0.00355	$0.00178 + j0.00178$	$0.00178 - j0.00178$
5	0.255	0	0.127	0.127

12.10 For the periodic function given in the Fourier series table in this chapter (the specific function provided by the instructor), determine or verify all forms of the series (trigonometric, amplitude/phase, and exponential) using the conversion relationships. Do not use the definitions for the Fourier series.

12.11C By using the properties of both linearity, combine two periodic functions (both provided by the instructor) given in the series table. Provide an analytical description of this new periodic function over one complete period. Plot this function using $n = 5$ and $n = 10$ terms.

12.12C For the fractional rectified cosine wave where $A = 2$, $T = 2$ μs, and $k = 0.2$, determine, starting from the Fourier series definitions, both the trigonometric and exponential Fourier series for the dc term and the first 5 nonzero coefficients. Then, compare the magnitude of the coefficients with the expression provided in the table. Finally, plot both the exact waveform and the truncated trigonometric and exponential series over two periods.

12.13C For the cosine pulse train where $A = 2$, $T = 2$ ms, and $\tau = 0.3$ ms, determine, starting from the Fourier series definitions, both the trigonometric and exponential Fourier series for the dc term and the first 5 nonzero coefficients. Then, compare the magnitude of the coefficients with the expression provided in the table. Finally, plot both the exact waveform and the truncated trigonometric and exponential series over two periods.

12.14C For the cosine-squared pulse train where $A = 2$, $T = 2$ ms, and $\tau = 0.3$ ms, determine, starting from the Fourier series definitions, both the trigonometric and exponential Fourier series for the dc term and the first 5 nonzero coefficients. Then, compare the magnitude of the coefficients with the expression provided in the table. Finally, plot both the exact waveform and the truncated trigonometric and exponential series over two periods.

12.15EC For the filter given in Figure 1, plot both the output response in the time domain and the frequency domain (i.e., amplitude and phase spectrums) for a periodic input voltage provided by your insightful instructor. Three different values of R should be used. After selecting a reasonable value for C, the three values for R should be such that (1) the lowest cutoff frequency is one-half of the fundamental frequency, (2) midway between the fundamental and the first harmonic frequency, and (3) ten times the first harmonic frequency. Then, comment on the time responses based on the speed and smoothness of the output.

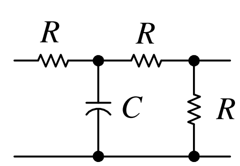

Figure 1

12.16EC Repeat Problem 12.15 for the circuit given in Figure 2.

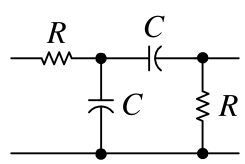

Figure 2

12.17 By comparing one specified Fourier transform pair (provided by the instructor) with two other transform pairs provided in the table in this chapter, provide two different partial checks of the Fourier transform pair. One or more of the transform properties given in the Fourier transform properties table must be used. These partial checks may not consist of merely multiplying the function or transform by a constant, setting a variable to zero, to infinity, or to another constant, or splitting or combining the function or transform (e.g., linearity). Do not use the definition for the Fourier transform or the inverse transform as a check.

12.18 For the Laplace transform pair number # given in the Laplace Transform table in this chapter (# provided by the instructor), determine whether all of the poles of the Laplace transform have negative real parts. If so, determine through $s = j\omega$ substitution, the corresponding Fourier transform.

12.19 For the two time-domain functions provided in the Laplace transform table in this book given as #1 and #2 (both functions #'s provided by the instructor), determine whether all of the poles of both corresponding Laplace transforms have negative real parts. If both of the transforms satisfy this criteria, sketch the time-domain functions $s(t)$ and $k(t)$:

$$s(t) = f_{\#1}(t)u(t) + f_{\#2}(-t)u(-t)$$
$$k(t) = f_{\#1}(-t)u(-t) + f_{\#2}(t)u(t)$$

Then, determine the Fourier transforms for both of these functions using the $s = j\omega$ substitution approach (and the time-reversal property).

12.20 Working in the time domain, verify that the total energy of each of the given signals is equal to $2a$:

$$r(t) = \begin{cases} 1 & -a < t < a \\ 0 & \text{elsewhere} \end{cases} \qquad x(t) = \begin{cases} \dfrac{t}{3a} + 1 & -3a < t < 0 \\ \dfrac{-t}{3a} + 1 & 0 < t < 3a \\ 0 & \text{elsewhere} \end{cases}$$

$$y(t) = \begin{cases} \dfrac{1}{2}\left[1 + \cos\left(\dfrac{3\pi}{8a}t\right)\right] & -\dfrac{8a}{3} < t < \dfrac{8a}{3} \\ 0 & \text{elsewhere} \end{cases} \qquad z(t) = \begin{cases} e^{\frac{t}{2a}} & t < 0 \\ e^{-\frac{t}{2a}} & t > 0 \end{cases}$$

12.21 Verify the Bode magnitude plot provided in this chapter for each of the following transforms including all slopes and cutoff frequencies. Also, what is the magnitude (in dB) of the transform for frequencies much less than the lowest cutoff frequency?

a) $R(\omega) = \dfrac{2\sin(a\omega)}{\omega}$

b) $X(\omega) = 3a\left[\dfrac{\sin\left(\dfrac{3a\omega}{2}\right)}{\dfrac{3a\omega}{2}}\right]^2$

c) $Y(\omega) = \dfrac{9\pi^2 \sin\left(\dfrac{8a\omega}{3}\right)}{\omega\left(9\pi^2 - 64a^2\omega^2\right)}$

d) $Z(\omega) = \dfrac{4a}{1+4a^2\omega^2}$

12.22EC Compare the amplitude spectrums of a rectangular pulse and a Gaussian pulse. Assume that the energy and the maximum amplitude of both signals are about the same. Determine and plot both the approximate straight-line Bode magnitude plots and the exact plots, and was done in this chapter, for both of these signals. Then, determine the percent energy versus ω on the same set of axes for these signals for frequencies ranging from 1/10 to 10 of the cutoff frequency of the rectangular pulse.

12.23C The results given in this chapter for the energy within the second cutoff frequency for a trapezoidal waveform are conservative. Instead of approximating the sinusoidal function as in the book, use a numerical program to determine the fractional amount of energy contained from 0 to $1/(\pi\tau_r)$ Hz. The fractional amount of energy should be plotted versus the rise time for a fixed pulse width. Allow the rise time to vary from one one-hundredth of the pulse width to its maximum possible value. What relationship between the pulse width and the rise time is required so that 90% of the energy is contained within the second corner frequency?

12.24C Using a numerical program, plot the magnitude spectrum (in dB) of an aperiodic trapezoidal waveform when the rise time is both equal to and not equal to the fall time. Select a reasonable rise time, τ_r, fall time, τ_f, and pulse width, τ. The break frequencies should be clearly evident. Is there a simple relationship for the break frequencies when the rise time is not equal to the fall time?

12.25C If I_o = 12 kA, η = 0.7, n = 2, τ_1= 0.2 μs, and τ_2 = 0.2 μs, determine the peak current, maximum current derivative, and total charge transfer if a lightning pulse is modeled using the Heidler function:

$$I(t) = \frac{I_o}{\eta} \frac{\left(\frac{t}{\tau_1}\right)^n}{\left(\frac{t}{\tau_1}\right)^n + 1} e^{-\frac{t}{\tau_2}}$$

12.26S For $n = 1$, determine or locate the Fourier transform of the Heidler function:

$$I(t) = \frac{I_o}{\eta} \frac{\left(\frac{t}{\tau_1}\right)^n}{\left(\frac{t}{\tau_1}\right)^n + 1} e^{-\frac{t}{\tau_2}}$$

Sketch the Bode magnitude plot of its amplitude spectrum. Compare this spectrum with the double-exponential's spectrum. [Rakov]

12.27C Using the plots given in this chapter for the double-exponential pulse, determine the values for α and β to model a lightning stroke with a rise time of 100 ns and pulse width of 60 μs. Then, adjust C so that the maximum amplitude of the stroke is 40 kA. Using these values, plot the resultant double-exponential function.

12.28 Sketch two different impulse time functions that have approximately the same 50% delay times but clearly different impulse responses. Then, sketch two similar impulse responses that have clearly different 50% delay times.

12.29 Sketch two different impulse time functions that have approximately the same centroid delay times but clearly different impulse responses. Then, sketch two similar impulse responses that have clearly different centroid delay times.

12.30 Sketch two different impulse time functions that have approximately the same 10-90% rise times but clearly different impulse responses. Then, sketch two similar impulse responses that have clearly different 10-90% rise times.

12.31 Sketch two different impulse time functions that have approximately the same maximum sloped-based rise times but clearly different impulse responses. Then, sketch two similar impulse responses that have clearly different maximum slope-based rise times.

12.32 Sketch two different impulse time functions that have approximately the same standard deviation rise times but different impulse responses. Then, sketch two similar impulse responses that have different standard-deviation rise times.

12.33 Verify that the average time delay for the rectangular pulse waveform

$$x(t) = u(t) - u(t-a)$$

is given by the expression

$$w(\tau_d) = w(\tau_d + a)$$

where $w(t)$ is the output response to the low pass system with a steady-state value of one; that is, the average time delay can be determined by examining the output response and determining the time where the output response is repeated a seconds later. [Blinchikoff]

12.34EC Determine the 10-90% rise time of an *RL* low-pass filter to a ramp input signal with a rise time of about τ_r. Compare to the perfect step input response.

12.35EC Determine the 10-90% rise time of an *RC* low-pass filter to a ramp input signal with a rise time of about τ_r. Compare to the perfect step input response.

12.36C For the Laplace transform pair number # given in the Laplace transform table in this book (# provided by the instructor), determine the centroid delay time and the standard deviation using the <u>series expansion</u> of the Laplace transform. Then, rewrite the transform so that it is in the form

$$H(s) = K\frac{1 + a_1 s + a_2 s^2 + \ldots + a_n s^n}{1 + b_1 s + b_2 s^2 + \ldots + b_m s^m}$$

where K is a constant. As a check on the delay and rise time results obtained from the series expansion, determine the delay time and the rise time from the expressions

$$\tau_d = b_1 - a_1,\ \tau_r = \sqrt{2\pi\left[b_1^2 - a_1^2 + 2(a_2 - b_2)\right]}$$

Assuming the given transform pair represents the Laplace transform of an impulse response, is the step response monotonic?

12.37C Starting from their definitions and working with the time-domain impulse and step response functions, verify each of the delay and rise times listed in the tables in this chapter for the

a) single-pole exponential pulse
b) double-pole critically damped pulse
c) causal Gaussian pulse
d) noncausal Gaussian pulse

12.38C Starting from the definitions, verify each of the bandwidths listed in the table in this chapter for the

a) single-pole exponential pulse, $H(s) = 1/(s+a)$

b) double-pole critically damped pulse, $H(s) = 1/(s+a)^2$

c) noncausal Gaussian pulse, $H(\omega) = a\sqrt{2\pi}e^{-\frac{a^2\omega^2}{2}}$

d) sampling function, $H(\omega) = 2\sin(\omega T/2)/\omega$

12.39 Using the slope-based rise time

$$\tau_r = \frac{\int_{-\infty}^{\infty} h(t)\,dt}{h(t_o)}$$

and a modified version of the equivalent bandwidth

$$BW_{eq} = \frac{\int_{-\infty}^{\infty} H(\omega)\,d\omega}{H(0)}$$

determine whether the time-bandwidth product is always 2π. [Soliman]

12.40 Verify each of the rise time-bandwidth products terms given in the matrix table in this chapter for the single-pole exponential pulse. Which of the products is the smallest? Which of the products is the largest? Qualitatively explain, without performing any calculations or without reference to this table, which of these products should be the smallest and largest.

12.41 Verify each of the rise time-bandwidth products terms given in the matrix table in this chapter for the Gaussian pulse. Which of the products is the smallest? Which of the products is the largest? Qualitatively explain, without performing any calculations or without reference to this table, which of these products should be the smallest and largest.