Kettering University Mathematics Olympiad For High School Students 2005, Sample Solutions

1. (Solution by Meelap Shah, a 4th-6th finisher)
   Expanding the left side, we get
   \[ 1 + x^2 + x^4 + x^6 = 4x^3 \Rightarrow (x^6 - 2x^3 + 1) + (x^4 - 2x^3 + x^2) = 0 \Rightarrow (x^3 - 1)^2 + x^2(x - 1)^2 = 0. \]
   Since all terms on the left hand side are non-negative, they both must be zero. Hence we have
   \[(x^3 - 1)^2 = 0 \Rightarrow x^3 = 1 \Rightarrow x = 1.\]
   and \[x^2(x - 1)^2 = 0 \Rightarrow x = 0 \text{ or } x = 1.\]
   So the only possible solution is \(x = 1\).

2. (Solution by Frederic Sala, a 4th-6th finisher)
   We show that since Nick goes first, there is a guaranteed strategy for him to win. Let Nick takes 1 pebble on his first turn, leaving 99. now, if John takes \(x\) pebbles (\(1 \leq x \leq 8\)), Nick will take \(9 - x\) pebbles. So for each pair of turns, 9 pebbles will be removed. After 10 turns, there will be \(99 - 9(10) = 0\) pebbles left. Whatever number, between 1 and 8, of pebbles John takes, Nick can take the remaining number and win.

   John can only be guaranteed to win if Nick doesn’t take 1 pebble on the first turn. If Nick takes between 2 and 8 pebbles, there will be between 92 and 98 pebbles left. John should take enough to be left with 90. Thereafter, John should take \(9 - x\), where \(x\) is the number of pebbles that Nick took. Again, we will be down to 9 pebbles when it is Nick’s turn, guaranteeing John a win. In general, either player should try to leave a multiple of 9 pebbles for his opponent. The first to do so wins by following the strategy above.

3. (Solution by Kevin Dilks, Mr Dilks placed third in the 2005 Kettering Math Olympiad.)
   First, note that \(\cos 12x = \cos^2 6x - \sin^2 6x = 1 - 2\sin^2 6x\). So we need to show that
   \[\sin x(\sin x + \sin 3x + \sin 5x + \cdots + \sin 11x) = \sin^2 6x.\]
   We see that
   \[\sin x = \sin(6x - 5x) = \sin 6x \cos 5x - \sin 5x \cos 6x\]
   \[\sin 11x = \sin(6x + 5x) = \sin 6x \cos 5x + \sin 5x \cos 6x\]
   \[\Rightarrow \sin x + \sin 11x = 2 \sin 6x \cos 5x.\]
   Similar we can be obtain the following identities:
   \[\sin 3x + \sin 9x = 2 \sin 6x \cos 3x,\]
   \[\sin 3x + \sin 9x = 2 \sin 6x \cos x.\]
Hence we need to show that
\[ 2 \sin x \sin 6x (\cos x + \cos 3x + \cos 5x) = \sin^2 6x. \]

Or equivalently, we need to show that
\[ 2 \sin x (\cos x + \cos 3x + \cos 5x) = \sin 6x. \]

\[
\cos 5x = \cos(4x + x) = \cos 4x \cos x - \sin 4x \sin x \\
\cos 3x = \cos(4x - x) = \cos 4x \cos x + \sin 4x \sin x \\
\Rightarrow \cos 3x + \cos 5x = 2 \cos 4x \cos x.
\]

So
\[ 2 \sin x \cos x (1 + 2 \cos 4x) = 2 \sin x \cos x [1 + 2(2 \cos^2 2x - 1)] = \sin 2x (4 \cos^2 2x - 1). \]

Working with the right hand side, we have
\[
\sin 6x = \sin (4x + 2x) = \sin 4x \cos 2x + \sin 2x \cos 4x \\
= 2 \sin 2x \cos^2 2x + \sin 2x \cos 4x \\
= \sin 2x (2 \cos^2 2x + \cos 4x) \\
= \sin 2x (2 \cos^2 2x + 2 \cos^2 2x - 1) \\
= \sin 2x (4 \cos^2 2x - 1)
\]

which equals the left hand side. Thus the equality holds.

4. (Solution by Bohao Pan, a 4th-6th finisher)
Nick starts off with 7 pieces. Suppose Nick cuts \( n_1 \) pieces. Then he now has:
\[ 7 - n_1 + 7n_1 = 7 + 6n_1 \]

In the next turn, Nick cuts \( n_2 \) pieces. He now has:
\[ (7 + 6n_1) - n_2 + 7n_2 = 7 + 6(n_1 + n_2) \]

In the \( k \)th turn, Nick cuts \( n_k \) pieces. He now has:
\[ [(7 + 6(n_1 + n_2 + \cdots + n_{k-1})) - n_k + 7n_k = 7 + 6(n_1 + n_2 + \cdots + n_k) \]

Nick claims that he now has 2000 pieces. That is,
\[ 7 + 6(n_1 + n_2 + \cdots + n_k) = 2000 \Rightarrow 6(n_1 + n_2 + \cdots + n_k) = 1993. \]

Since 1993 is not divisible by 6, no integer values \( n_1, n_2, \ldots, n_k \) satisfies the above. Thus John concludes that Nick made an error.
5. (Sample Solution:) From the shown diagram, we see that

\[
\text{Area}(ACM) = \text{Area}(AMD)
\]

\[
\Rightarrow \frac{1}{2}|AC||AM| \sin(3a) = \frac{1}{2}|AM||AD| \sin(a)
\]

\[
\Rightarrow |AC| \sin(3a) = |AD| \sin(a)
\]

\[
\Rightarrow |AD| = \frac{\sin(3a)}{\sin(a)} |AC|
\]

Also

\[
|AH| = |AC| \cos(a) = |AD| \cos(3a)
\]

\[
\Rightarrow |AD| = \frac{\cos(a)}{\cos(3a)} |AC|
\]

\[
\Rightarrow \frac{\sin(3a)}{\sin(a)} = \frac{\cos(a)}{\cos(3a)}
\]

\[
\Rightarrow \sin(3a) \cos(3a) - \sin(a) \cos(a) = 0
\]

\[
\Rightarrow \frac{1}{2} \sin(6a) - \frac{1}{2} \sin(2a) = 0
\]

\[
\Rightarrow \frac{1}{2} \left( \sin(2a) \cos(4a) + \sin(4a) \cos(2a) - \sin(2a) \cos(2a) \right) = 0
\]

\[
\Rightarrow \frac{1}{2} \left( \sin(2a) \cos(4a) + 2 \sin(2a) \cos^2(2a) - \sin(2a) \right) = 0
\]

\[
\Rightarrow \sin(2a) \cos(4a) = 0
\]

where \(0 < 4a < \pi \Rightarrow 0 < a < \frac{\pi}{4} \Rightarrow \sin(2a) \neq 0\). Hence \(\cos(4a) = 0 \Rightarrow 4a = \frac{\pi}{2} \Rightarrow a = \frac{\pi}{8}\).

Figure 1: Diagram For Question 5
6. (a) (*Solution by Alex Xu, Mr Xu placed second in the 2005 Kettering Math Olympiad.*)

First every city must contain a connection. Since two cities require one line to be connected, adding a city adds a minimum of one line in order for the new city to be connected. Thus, for \( n \) cities, a minimum of \( n - 1 \) connectors is needed just to connect all of the cities.

For \( k = 1 \), each city must be connected to every other city. Thus \( \binom{100}{2} = \frac{(100)(99)}{2} = 4950 \) airlines are needed. For \( k = 2 \), each city can be connected to a single city. This uses \( n - 1 \) connections, which is the minimum number of connections. This setup also satisfies all \( k > 2 \).

One possible scheme is to connect city 1 to every other city. This gives 99 connections.

(b) (Sample Solution:) If all cities are connected to each other, there is nothing to prove. So let’s divide the 100 cities into two distinct groups, A and B respectively, where no cities in group A is connected to cities in group B. Let the number of cities in group A be \( k \), then the number of cities in group B is \( 100 - k \). The total number of airlines needed to ensure that all cities in group A is connected directly is:

\[
(k - 1) + (k - 2) + \cdots + 1 = \frac{k(k - 1)}{2}.
\]

Similarly, The total number of airlines needed to ensure that all cities in group B is connected directly is:

\[
(99 - k) + (98 - k) + \cdots + 1 = \frac{(99 - k)(100 - k)}{2}.
\]

The total number of airlines needed thus far is:

\[
T(k) = \frac{k(k - 1)}{2} + \frac{(99 - k)(100 - k)}{2} = k^2 - 100k + 4950 = (k - 50)^2 + 2450, \quad 1 \leq k \leq 99.
\]

The above parabola attains its maximum in the allowable range of \( k \) at \( k = 1 \) and \( k = 99 \) with \( T(1) = T(99) = 4851 \). So with 4852 airlines we can use the remaining airline to connect any city in group A to group B.