With Cartesian coordinates \((x, y)\) of a point \(P\),

a) \(x\) denotes the distance from the \(y\) axis to \(P\), and
b) \(y\) denotes the distance from the \(x\) axis to \(P\).

We can define an alternative coordinate system, called **polar coordinates**, in the following manner.

1) Define the ray \(OP\) from the origin \(O\) to \(P\).
2) Let \(r\) denote the length of ray \(OP\).
3) Let \(\theta\) denote the angle from the \(+x\) axis to ray \(OP\).  
   The \(+x\) axis is called the **polar axis**.
Angles measured counterclockwise from the polar axis are called positive. Angles measured clockwise from the polar axis are called negative.

4) Then the polar coordinates of point $P$ are $(r, \theta)$.

In the above example, $r = \sqrt{x^2 + y^2} = \sqrt{(4)^2 + (3)^2} = 5$.

The angle $\theta$ is determined by

$$\tan \theta = \frac{y}{x} = \frac{3}{4} \Rightarrow \theta = \tan^{-1}\left(\frac{3}{4}\right) = 0.6425 \text{ rad} = 36.87^\circ.$$ 

So point $P$ has polar coordinates $(r, \theta) = (5, 0.6425) = (5, 36.87^\circ)$.

In general then, if a point $P$ has Cartesian coordinates $(x, y)$, then its polar coordinates $(r, \theta)$ are determined by the transformation equations:

$$r^2 = x^2 + y^2, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

(use these to convert Cartesian coords to polar coords)
We define the origin in polar coordinates, called the **pole**, by

\[
( r, \theta ) = (0, \theta ) \quad \text{for any angle } \theta .
\]

Good polar graph paper would look like

\[
x = r \cos \theta , \quad y = r \sin \theta . \quad \text{(use these to convert polar coords to Cartesian coords)}
\]

Here's the point \((x, y) = (4, 3)\), i.e. \((r, \theta) = (5, 0.6425)\) plotted in polar coordinates:
Note: We've merely described the point $P$ in space using different coordinate systems.

Curves in Polar Coordinates

A curve in polar coordinates is usually represented by a formula of the form

$$r = f(\theta).$$

Example: The simple expression $r = 5$ is independent of $\theta$, so it represents the set of all points that are 5 units from the origin, i.e., a circle of radius 5 centered at $O$. 
In general, an expression \( r = a \) represents a circle of radius \( a \) centered at \( O \).

**Example:** The simple expression \( \theta = \frac{\pi}{3} \) is independent of \( r \), so it represents the straight line passing through \( O \) that makes an angle of \( \frac{\pi}{3} \) radians relative to the polar axis (the \( +x \) axis).
Note: If the radial coordinate $r$ is negative, e.g., the point $(r, \theta) = \left(-5, \frac{\pi}{4}\right)$, then the point is along the ray $\theta = \frac{\pi}{4}$ but 5 units in the opposite direction.
Therefore, the polar representation of a point is not unique:

\[(r, \theta) = \left(-5, \frac{\pi}{4}\right) = \left(+5, \frac{5\pi}{4}\right) = \left(+5, -\frac{3\pi}{4}\right) = \left(-5, \frac{9\pi}{4}\right) = \ldots\]

**Example:** What is the polar representation of the circle of radius \(a\) centered at the point \((x, y) = (a, 0)\)?

\[(x - a)^2 + y^2 = a^2\]
where \( x = r \cos \theta \) and \( y = r \sin \theta \). Substitute into the above:

\[
\begin{align*}
\Rightarrow \quad & r^2 \cos^2 \theta - 2a r \cos \theta + r^2 \sin^2 \theta = 0 \\
\Rightarrow \quad & r^2 (\cos^2 \theta + \sin^2 \theta) = 2a r \cos \theta \\
\Rightarrow \quad & r = 2a \cos \theta.
\end{align*}
\]

So the polar representation of the circle of radius \( a \) centered at the point \((x, y) = (a, 0)\) is

\[ r = 2a \cos \theta, \quad 0 \leq \theta \leq \pi. \]

In a similar way, the polar representation of the circle of radius \( a \) centered at the point \((x, y) = (0, a)\) is

\[ r = 2a \sin \theta, \quad 0 \leq \theta \leq \pi. \]

**Example:** The circle of radius 3 centered at the point \((x, y) = (3, 0)\) is (blue), and the circle of radius 3 centered at the point \((x, y) = (0, 3)\) is (green),

\[ r = 6 \cos \theta, \quad r = 6 \sin \theta. \]
$r = 6 \cos \theta, \quad r = 6 \sin \theta$

```
xanim:=animatecurve([6*cos(theta), theta, theta=0..Pi], coords=polar, color=blue, axiscoordinates=polar, frames=50, thickness=6, numpoints=200, scaling=constrained):

yanim:=animatecurve([6*sin(theta), theta, theta=0..Pi], coords=polar, color=green, axiscoordinates=polar, frames=50, thickness=6, numpoints=200, scaling=constrained):
```
display([xanim, yanim]);

\[ r = 6 \cos \theta, \quad r = 6 \sin \theta \]

Notice that each circle makes one revolution counterclockwise as \( \theta \) varies from 0 to \( \pi \).

**Example:** What do the equations represent?

\[ a) \quad r = -6 \cos \theta, \quad b) \quad r = -6 \sin \theta. \]
They are:
   a)  a circle of radius 3 centered at \((x, y) = (-3, 0)\) and
   b)  a circle of radius 3 centered at \((x, y) = (0, -3)\).

> xanim:=animatecurve( [-6*cos(theta), theta,theta=0..Pi],coords=polar,color=blue,axiscoordinates=polar,frames=50,thickness=6, numpoints=200,scaling=constrained ):  

> yanim:=animatecurve( [-6*sin(theta), theta,theta=0..Pi],coords=polar,color=green,axiscoordinates=polar,frames=50,thickness=6, numpoints=200,scaling=constrained ):  

> display([xanim, yanim]);
Symmetry in Polar Coordinates

1. If a polar equation is unchanged when $\theta$ is replaced by $-\theta$, then the curve is symmetric about the polar axis ($x$ axis). See the example $r = 6 \cos \theta$: since
\[ r = 6 \cos(-\theta) = 6 \cos \theta. \]

2. If a polar equation is unchanged when $r$ is replaced by $-r$ OR $\theta$ is replaced by $\theta + \pi$, then the curve is symmetric about the pole (origin). This means the curve remains unchanged if it is rotated $180^\circ$ about the origin.
3. If a polar equation is unchanged when $\theta$ is replaced by $\pi - \theta$, then the curve is symmetric about the line $\theta = \frac{\pi}{2}$ (y axis). See the example $r = 6 \sin \theta$: since

$$r = 6 \sin(\pi - \theta) = 6 \sin(-\theta + \pi) = -6 \sin(-\theta) = 6 \sin(\theta).$$

**Limaçons**

Consider a polar equation of the form

$$r = b + c \cos \theta,$$

for constants $b$ and $c$.

For example, consider the limaçon

$$r = 1 + \frac{3}{4} \cos \theta.$$

```latex
> animatecurve([1+3/4*cos(theta), theta, theta = 0..2*Pi], coords=polar, color=blue, axiscoordinates=polar, frames=81, thickness=6, numpoints=200, scaling=constrained);
```
Now consider the limaçon: 

\[ r = 1 + \frac{3}{4} \cos \theta \]

Now consider the limaçon: \( r = 1 + \cos \theta \).

\[
> \text{animatecurve( [1+cos(theta), theta,theta = 0..2*Pi],coords=polar, color=blue,axiscoordinates=polar,frames=81, thickness=6, numpoints=200,scaling=constrained);}\]
This limaçon has a cusp at the pole. This curve is called a **cardioid**.

Now consider the limaçon: \[ r = 1 + 2 \cos \theta \]

This limaçon has a cusp at the pole. This curve is called a **cardioid**.

Now consider the limaçon: \[ r = 1 + 2 \cos \theta \].
This limaçon has a loop.

If fact,
- if $c < 1$, the limaçon has a smooth dimple,
- if $c = 1$, the limaçon has a cusp (and the curve is called a cardioid),
- if $c > 1$, the limaçon crosses itself and has a loop.

Now let's consider

$$r = 1 + 2 \cos \theta$$
\[ r = 1 + c \cos \theta \]

as we increase the \( c \) value of from \( c=0 \) to \( c=4 \).
Likewise, consider

\[ r = 1 + c \cos \theta \]

as we increase the \( c \) value of from \( c=0 \) to \( c=4 \).

\[
> \text{animate(plot, [1+c*sin(theta), theta = 0 .. 2*Pi,coords=polar,}
\]

- -
\text{axiscoordinates=polar, color=blue, thickness=6, numpoints=200, scaling = constrained], c=0..4,frames=81 );

\[ c = 4.0000 \]

\[ r = 1 + c \sin \theta \]

Roses

\[ r = \cos n\theta \quad \text{and} \quad r = \sin n\theta \]

where \( n \) is an integer.
• If $n$ is even, then the rose has $2n$ petals on interval $0 \leq \theta \leq 2\pi$.
• If $n$ is odd, then the rose has $n$ petals on interval $0 \leq \theta \leq \pi$.

Example:

Graph the rose $r = \cos 2\theta$. Since $n = 2$ (even), the rose as 4 petals on $0 \leq \theta \leq 2\pi$.

> animatecurve( [cos(2*theta), theta, theta = 0 .. 2*Pi], coords=polar, axiscoordinates=polar, frames=81, color=blue, thickness=6, numpoints=200, scaling = constrained );
The top half of the 1st petal (on the $+x$ axis) occurs when $0 \leq \theta \leq \frac{\pi}{4}$.

The 2nd petal (on the $-y$ axis) occurs when $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$.

The 3rd petal (on the $-x$ axis) occurs when $\frac{3\pi}{2} \leq \theta \leq \frac{5\pi}{4}$.

The 4th petal (on the $+y$ axis) occurs when $\frac{5\pi}{4} \leq \theta \leq \frac{7\pi}{4}$.

The bottom half of the 1st petal (on the $+x$ axis) occurs when $\frac{7\pi}{4} \leq \theta \leq 2\pi$.

Example:

Graph the rose $r = \cos 3\theta$. Since $n = 3$ (odd), the rose as 3 petals on $0 \leq \theta \leq \pi$.

> animatecurve( [cos(3*theta), theta, theta = 0 .. Pi], coords=polar, axiscoordinates=polar, frames=81, color=blue, thickness=6, numpoints=200, scaling = constrained );
The top half of the 1st petal (on the $+x$ axis) occurs when $0 \leq \theta \leq \frac{\pi}{6}$.

The 2nd petal (in Quad III along $\theta = \frac{2\pi}{6} = \frac{\pi}{3}$) occurs when $\frac{\pi}{6} \leq \theta \leq \frac{3\pi}{6}$.

The 3rd petal (In Quad II along $\theta = \frac{4\pi}{6} = \frac{2\pi}{3}$) occurs when $\frac{3\pi}{6} \leq \theta \leq \frac{5\pi}{6}$.

The bottom half of the 1st petal (on the $+x$ axis) occurs when $\frac{5\pi}{6} \leq \theta \leq \pi$. 

$r = \cos 3\theta$
Example:

Graph the rose $r = \sin 2\theta$. Since $n = 2$ (even), the rose as 4 petals on $0 \leq \theta \leq 2\pi$.

```maple
> animatecurve([sin(2*theta), theta, theta = 0..2*Pi], coords=polar, color=blue, axiscoordinates=polar, frames=81, thickness=6, numpoints=200, scaling=constrained);
```

$r = \sin 2\theta$

The 1st petal (in Quad I) occurs when $0 \leq \theta \leq \frac{\pi}{2}$.

The 2nd petal (in Quad IV) occurs when $\frac{\pi}{2} \leq \theta \leq \pi$. 
The 3rd petal (in Quad III) occurs when \( \pi \leq \theta \leq \frac{3\pi}{2} \).

The 4th petal (in Quad II) occurs when \( \frac{3\pi}{2} \leq \theta \leq 2\pi \).

**Example:**

Graph the rose \( r = \sin 3\theta \). Since \( n = 3 \) (odd), the rose as 3 petals on \( 0 \leq \theta \leq \pi \).

\[
\text{animatecurve( [sin(3*theta), theta,theta = 0..Pi],coords=polar, color=blue,axiscoordinates=polar,frames=81,thickness=6,numpoints=200,scaling=constrained);}
\]
Suppose we have a curve in polar coordinates given in the form.

The 1st petal (in Quad I along $\theta = \frac{\pi}{6}$) occurs when $0 \leq \theta \leq \frac{\pi}{3}$.

The 2nd petal (along the $-y$ axis) occurs when $\frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3}$.

The 3rd petal (in Quad II along $\theta = \frac{5\pi}{6}$) occurs when $\frac{2\pi}{3} \leq \theta \leq \pi$.

**Slope of a Tangent Line in Polar Coordinates**

Suppose we have a curve in polar coordinates given in the form

$$r = \sin 3 \theta$$
Recall that the slope of the tangent line to a curve is $\frac{dy}{dx} = \frac{d\theta}{dx}$, where recall

\[ x = r \cos \theta = f(\theta) \cos \theta \quad \text{and} \quad y = r \sin \theta = f(\theta) \sin \theta. \]

So

\[ \frac{dx}{d\theta} = f'(\theta) \cos \theta - f(\theta) \sin \theta \quad \text{and} \quad \frac{dy}{d\theta} = f'(\theta) \sin \theta + f(\theta) \cos \theta. \]

Therefore,

\[ \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}. \]

**Example:** Determine where the cardioid has horizontal and vertical tangent lines.

\[ r = 1 + \sin \theta = f(\theta), \quad 0 \leq \theta \leq 2\pi. \]

Here $f(\theta) = 1 + \sin \theta$, so $f'(\theta) = \cos \theta$. So

\[ \frac{dy}{dx} = \frac{\cos \theta \sin \theta + (1 + \sin \theta) \cos \theta}{\cos^2 \theta - (1 + \sin \theta) \sin \theta} = \frac{\cos \theta (1 + 2 \sin \theta)}{1 - \sin^2 \theta - \sin \theta - \sin^2 \theta} = \frac{\cos \theta (1 + 2 \sin \theta)}{1 - \sin \theta - 2 \sin^2 \theta} = \frac{\cos \theta (1 + 2 \sin \theta)}{(1 + \sin \theta) (1 - 2 \sin \theta)}. \]

a) The numerator is 0 if $\cos \theta = 0$ or $\sin \theta = -\frac{1}{2}$:

\[ \theta = \frac{\pi}{2}, \quad \frac{3\pi}{2}, \quad \text{and} \quad \theta = \frac{7\pi}{6}, \quad \frac{11\pi}{6}. \]

So the cardioid has horizontal tangent lines at the points with polar coordinates

\[ (r, \theta) = \left(2, \frac{\pi}{2}\right), \quad \left(2, \frac{3\pi}{2}\right), \quad \left(\frac{1}{2}, \frac{7\pi}{6}\right), \quad \left(\frac{1}{2}, \frac{11\pi}{6}\right). \]

b) The denominator is 0 if $\sin \theta = -1$ or $\sin \theta = \frac{1}{2}$:

\[ \theta = \frac{3\pi}{2}, \quad \text{and} \quad \theta = \frac{\pi}{6}, \quad \frac{5\pi}{6}. \]

**NOTE:** Since both numerator and denominator are 0 at $\theta = \frac{3\pi}{2}$, we must use L'Hospital's rule to
determine the slope of the tangent line at $\theta = \frac{3\pi}{2}$.

a) The numerator is 0 when:
$$\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}.$$ So the cardioid has horizontal tangent lines at the points with polar coordinates
$$(r, \theta) = \left(2, \frac{\pi}{2}\right), \left(\frac{1}{2}, \frac{7\pi}{6}\right), \left(\frac{1}{2}, \frac{11\pi}{6}\right).$$

b) The denominator is 0 when:
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$ So the cardioid has vertical tangent lines at the points with polar coordinates
$$(r, \theta) = \left(\frac{3}{2}, \frac{\pi}{6}\right), \left(\frac{3}{2}, \frac{5\pi}{6}\right).$$

Now we use L'Hospital's rule to determine the slope of the tangent line at $\theta = \frac{3\pi}{2}$.

$$\lim_{\theta \to \frac{3\pi}{2}} \frac{dy}{dx} = \lim_{\theta \to \frac{3\pi}{2}} \frac{\cos \theta + 2 \sin \theta \cos \theta}{1 - \sin \theta - 2 \sin^2 \theta} \quad \text{trig identity: } \sin 2\theta = 2 \sin \theta \cos \theta$$

$$= \lim_{\theta \to \frac{3\pi}{2}} \frac{\cos \theta + \sin 2\theta}{1 - \sin \theta - 2 \sin^2 \theta} \sim \frac{0}{0}$$

$$= \lim_{\theta \to \frac{3\pi}{2}} \frac{-\sin \theta + 2 \cos 2\theta}{-\cos \theta - 4 \sin \theta \cos \theta} \sim \frac{0}{0} \quad \text{by L'Hospital's rule}$$

$$= \lim_{\theta \to \frac{3\pi}{2}} \frac{-\sin \theta + 2 \cos 2\theta}{-\cos \theta - 2 \sin 2\theta} \quad \text{since } \sin 2\theta = 2 \sin \theta \cos \theta$$

$$= \lim_{\theta \to \frac{3\pi}{2}} \frac{-\sin \theta + 2 \cos 2\theta}{-\cos \theta - 2 \sin 2\theta} \quad \text{since } \sin 2\theta = 2 \sin \theta \cos \theta$$

$$= \lim_{\theta \to \frac{3\pi}{2}} \frac{-\sin \theta + 2 \cos 2\theta}{-\cos \theta - 2 \sin 2\theta} \quad \text{since } \sin 2\theta = 2 \sin \theta \cos \theta$$

$$= \frac{-(-1) + 2 \cdot 0}{0 - 2 \cdot 0}$$

$$= \frac{1}{0} = \infty.$$ So the cardioid also has a vertical tangent line at the point
$$(r, \theta) = \left(0, \frac{3\pi}{2}\right) = (0, 0), \text{ since this is the origin (the pole)}.$$