Inverse Matrix, Norms, and Condition Numbers
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files: ConditionNumber.mw, ConditionNumber.pdf
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First load the LinearAlgebra library, and create a square matrix \( A \):

\[
\text{with(LinearAlgebra) :}
\]
\[
A := \text{Matrix}([[0.835, 0.667], [0.333, 0.266]])
\]

\( A := \begin{bmatrix} 0.835 & 0.667 \\ 0.333 & 0.266 \end{bmatrix} \) (1)

Calculate the determinant of \( A \) and its inverse, which we'll call \( A^{-1} \). We'll confirm the inverse by multiplying \( A \) and \( A^{-1} \).

\[
\text{Determinant}(A)
\]

# Notice that the determinant is very small. This suggests that matrix \( A \) is ill-conditioned.
\[
-0.000001 \quad (2)
\]

\[
AI := \text{MatrixInverse}(A) \quad # \text{This is } A^{-1}, \text{except for possible truncation error.}
\]

\[
AI := \begin{bmatrix} -2.65999999993532 \times 10^5 & 6.66999999983782 \times 10^5 \\ 3.3299999991903 \times 10^5 & -8.3499999979697 \times 10^5 \end{bmatrix} \quad (3)
\]

\[
\text{Multiply}(A, AI) \quad # \text{This should produce the } 2 \times 2 \text{ identity matrix } I, \text{except for possible truncation error.}
\]

\[
\begin{bmatrix} 0.99999999970896 & 1.16415321826935 \times 10^{-10} \\ 0. & 1. \end{bmatrix} \quad (4)
\]

Define a constant vector \( b_1 \), and use LinearSolve to solve the system \( Ax = b_1 \). We'll call the solution \( x_1 \):

\[
b_1 := \text{Vector}([0.168, 0.067])
\]

\[
b_1 := \begin{bmatrix} 0.168 \\ 0.067 \end{bmatrix} \quad (5)
\]

\[
x_1 := \text{LinearSolve}(A, b_1) \quad # \text{This is the solution of system } Ax = b_1.
\]

\[
x_1 := \begin{bmatrix} 0.99999999972231 \\ -0.99999999965236 \end{bmatrix} \quad (6)
\]

To confirm that \( x_1 \) is the solution, we'll multiply \( A \) and \( x_1 \). We should get the constant vector \( b_1 \) except
for truncation error.

\[
> \text{Multiply}(A, x1)
\begin{bmatrix}
0.16800000000000000 \\
0.06700000000000000
\end{bmatrix}
\]

(7)

As further confirmation, we'll multiply \(A^{-1}\) and \(b1\). We should get the solution vector \(x1\) except for truncation error.

\[
> \text{soln} := \text{Multiply}(A1, b1)
\begin{bmatrix}
0.999999999978172 \\
-0.999999999978172
\end{bmatrix}
\]

(8)

Let's now define a slightly different constant vector \(b2\) and solve the system \(Ax = b2\). We'll call the solution \(x2\):

\[
> \text{b2} := \text{Vector}(\{0.167, 0.067\})
\begin{bmatrix}
0.167 \\
0.067
\end{bmatrix}
\]

(9)

\[
> x2 := \text{LinearSolve}(A, b2) \quad \# \text{This is the solution of system } Ax = b2.
\]

\[
\begin{bmatrix}
266.999999993511 \\
-333.99999991877
\end{bmatrix}
\]

(10)

To confirm that \(x2\) is the solution, we'll multiply \(A\) and \(x2\). We should get the constant vector \(b2\) except for truncation error.

\[
> \text{Multiply}(A, x2)
\begin{bmatrix}
0.167000000000030 \\
0.067000000000073
\end{bmatrix}
\]

(11)

As further confirmation, we'll multiply \(A^{-1}\) and \(b2\). We should get the solution vector \(x2\) except for truncation error.

\[
> \text{soln} := \text{Multiply}(A1, b2)
\begin{bmatrix}
266.999999993510 \\
-333.99999991880
\end{bmatrix}
\]

(12)

In the preceding examples, we solved systems \(Ax = b1\) and \(Ax = b2\) where the input vectors \(b1\) and \(b2\) differ by only one digit in the first component. Compare \(b1\) and \(b2\) again:

\[
> b1; b2
\]
That very small change to the input vector produced a VERY LARGE change in the solution $x$. Compare solutions $x_1$ and $x_2$ again:

$> x_1; x_2$

\[
\begin{bmatrix}
0.168 \\
0.067 \\
0.167 \\
0.067
\end{bmatrix}
\]

Why did a very small change to the input vector produce a very large change in the solution? Because the coefficient matrix $A$ is ill-conditioned. Recall that a matrix is ill-conditioned if its condition number is much larger than 1. To see this, we'll calculate the condition number of $A$ using the infinity norm.

\[
\begin{aligned}
\text{Recall that the condition number of } A & \text{ is defined by } \| A \| \cdot \| A^{-1} \|. \text{ Let's verify this:} \\
\end{aligned}
\]

$> \text{ConditionNumber}(A, \text{infinity})$  
\[
\begin{aligned}
\text{Note that the condition number of } A & \text{ is much larger than 1.} \\
1.754336000 \times 10^6
\end{aligned}
\]

Recall that the condition number of $A$ is defined by $\| A \| \cdot \| A^{-1} \|$. Let's verify this:

$> ANorm := \text{Norm}(A, \text{infinity})$  
\[
\begin{aligned}
\text{This is the infinity norm of } A. \\
ANorm := 1.502
\end{aligned}
\]

$> AINorm := \text{Norm}(A^{-1}, \text{infinity})$  
\[
\begin{aligned}
\text{This is the infinity norm of } A^{-1}. \\
AINorm := 1.1679999997160025 \times 10^6
\end{aligned}
\]

$> ANorm \cdot AINorm$  
\[
\begin{aligned}
\text{This should equal the condition number of } A \text{ that we obtained above.} \\
1.754336000 \times 10^6
\end{aligned}
\]

Let's now calculate the condition number of $A$ using the Frobenius norm.

$> \text{ConditionNumber}(A, \text{Frobenius})$  
\[
\begin{aligned}
\text{Note that the condition number of } A & \text{ is much larger than 1.} \\
1.323759000 \times 10^6
\end{aligned}
\]

$> ANorm := \text{Norm}(A, \text{Frobenius})$  
\[
\begin{aligned}
\text{This is the Frobenius norm of } A. \\
ANorm := 1.150547261
\end{aligned}
\]

$> AINorm := \text{Norm}(A^{-1}, \text{Frobenius})$  
\[
\begin{aligned}
\text{This is the Frobenius norm of } A^{-1}. \\
AINorm := 1.15054726106128586 \times 10^6
\end{aligned}
\]

$> ANorm \cdot AINorm$  
\[
\begin{aligned}
\text{This should equal the condition number of } A \text{ that we obtained above.} \\
1.323759000 \times 10^6
\end{aligned}
\]