Mathematical Origami:
PHiZZ Dodecahedron

We will describe how to make a *regular dodecahedron* using Tom Hull's *PHiZZ* modular origami units. First we need to know how many faces, edges and vertices a dodecahedron has. Let's begin by discussing the Platonic solids.

The Platonic Solids

A *Platonic solid* is a convex polyhedron with congruent regular polygon faces and the same number of faces meeting at each vertex. There are five Platonic solids: *tetrahedron, cube, octahedron, dodecahedron,* and *icosahedron*.

The solids are named after the ancient Greek philosopher Plato who equated them with the four classical elements: earth with the cube, air with the octahedron, water with the icosahedron, and fire with the tetrahedron). The fifth solid, the dodecahedron, was believed to be used to make the heavens.

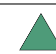
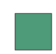
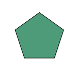
Why Are There Exactly Five Platonic Solids? Let's consider the vertex of a Platonic solid. Recall that the same number of faces meet at each vertex. Then the following must be true.

- There are at least three faces meeting at each vertex.
- The sum of the angles at a vertex must be less than 360 degrees. (Otherwise we would not have a convex polyhedron.)
- It follows that each interior angle of a polygon face must measure less than 120 degrees.
- The only regular polygons with interior angles measuring less than 120 degrees are the equilateral triangle, square and regular pentagon.

Therefore each vertex of a Platonic solid may be composed of

- 3 equilateral triangles (angle sum: $3 \times 60° = 180°$)
- 4 equilateral triangles (angle sum: $4 \times 60° = 240°$)
- 5 equilateral triangles (angle sum: $5 \times 60° = 300°$)
- 3 squares (angle sum: $3 \times 90° = 270°$)
- 3 regular pentagons (angle sum: $3 \times 108° = 324°$)

These are the five Platonic solids.

Solid	Face	# Faces/ Vertex	# Faces
tetrahedron		3	4
octahedron		4	8
icosahedron		5	20
cube		3	6
dodecahedron		3	12

How Many Vertices and Edges Does Each Solid Have?

Consider the dodecahedron. To calculate the number of vertices in a dodecahedron, we note that the solid is composed of 12 regular pentagons. If the pentagons were all separate, then we would have a total of $12 \times 5 = 60$ vertices. But 3 pentagons meet at each vertex so the number of vertices in a dodecahedron is

$$\begin{aligned} V &= \frac{(\#\text{ faces}) \times (\#\text{ vertices per face})}{(\#\text{ faces per vertex})} \\ &= \frac{12 \times 5}{3} = \frac{60}{3} = 20. \end{aligned}$$

We can calculate the number of vertices in the other Platonic solids using the same method.

To calculate the number of edges in a dodecahedron, we note that 12 regular pentagons have a total of $12 \times 5 = 60$ edges. But when we join the pentagons to make a dodecahedron, each edge meets another edge so the number of edges in a dodecahedron is

$$\begin{aligned} E &= \frac{(\#\text{ faces}) \times (\#\text{ edges per face})}{2} \\ &= \frac{12 \times 5}{2} = \frac{60}{2} = 30. \end{aligned}$$

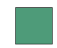
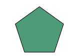
Similarly we can calculate the number of edges in the other Platonic solids.

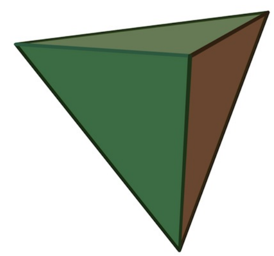
Euler's Formula for Polyhedra

We can check our answers using Euler's formula for convex polyhedra:

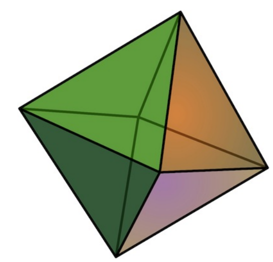
$$V - E + F = 2.$$

For each solid, the number of vertices minus the number of edges plus the number of faces equals 2.

Solid	Face	# Faces/ Vertex	# Faces	# Vertices	# Edges
tetrahedron		3	4	4	6
octahedron		4	8	6	12
icosahedron		5	20	12	30
cube		3	6	8	12
dodecahedron		3	12	20	30



tetrahedron



octahedron

Polyhedron Duals

Every Platonic solid has a *dual* polyhedron which is another Platonic solid. The dual is formed by placing a vertex in the center of each face of a Platonic solid. The resulting polyhedron is another Platonic solid. The dual of the octahedron is the cube and vice versa. Similarly the icosahedron and the dodecahedron are a dual pair. The tetrahedron is *self-dual*: its dual is another tetrahedron.

Making a PHiZZ Dodecahedron

Now that we know a dodecahedron is composed of 12 pentagon faces and a total of 30 edges, we are ready to make a dodecahedron out of PHiZZ modular origami units. Each PHiZZ unit will form one edge of the dodecahedron so we will need 30 square pieces of paper. (The 3" × 3" memo cube paper from *Staples* works well. Do not use sticky paper like Post-its.)

Refer to the *Pentagon-Hexagon Zig-Zag (PHiZZ) Unit* webpage for instructions on how to fold and join PHiZZ units.

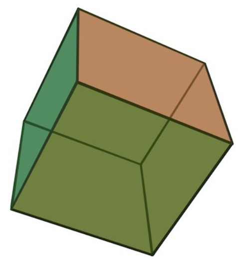
- `http://mars.wnec.edu/~thull/phzig/phzig.html`

Video demonstrations can be found at the following links.

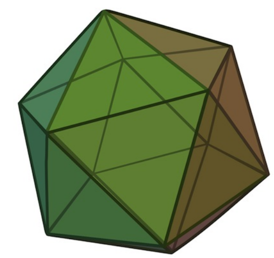
- PHiZZ Unit Part 1: `http://www.youtube.com/watch?v=vFYw47Wx2N8`
- PHiZZ Unit Part 2: `http://www.youtube.com/watch?v=dH-uTRdI4XU`

1. Begin by folding 3 PHiZZ units. Join the 3 edges to make one vertex of the dodecahedron.
2. Now add 7 more PHiZZ units to make a regular pentagon. (The extra units are needed to lock the vertices in place.)
3. Add to the pentagon until you have a complete dodecahedron.

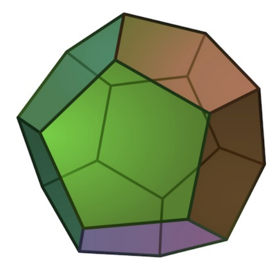
PHiZZ units can be used to make various polyhedra composed of pentagon and hexagon faces. For example, one can make a truncated icosahedron (aka soccer ball) using 90 PHiZZ units. Try it!



cube



icosahedron



dodecahedron

References

Hull, Tom. *Project Origami*. A K Peters Ltd, 2006.

"Platonic Solid." *Wikipedia, The Free Encyclopedia.* Wikimedia Foundation, Inc. 16 February 2011.

Pentagon-Hexagon Zig-Zag (PHiZZ) Unit

Over the past several years I've been playing with a certain member of the Zig-Zag family of modular origami units. **Zig-Zag units** are modular origami folds whose locking mechanism is based on an accordion pleat. That is, you accordion pleat a square, typically into 4ths or 3rds, making a rectangle. The short ends of the rectangle become the flaps, and the layers at the sides created by the accordion folds become the pockets. Of course, other folds in the rectangle are needed to make the flaps hook and stay, but that's the basic concept. Lots of modular origami units exist that fall into this category, units by creators such as Robert Neale, Lewis Simon, Jeannine Mosely, and Jun Maekawa, just to name a few.

The unit I've been using extensively is what I call the **pentagon-hexagon zig-zag unit** (or **PHiZZ unit**). I call it this because you can use it to make any polyhedron that

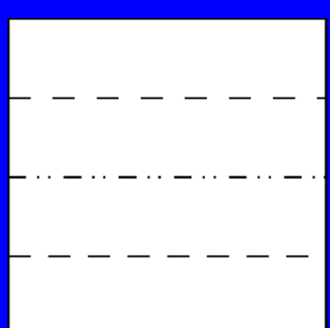
1. is cubic - each corner of the polyhedron has three edges meeting it, and
2. has only pentagon and hexagon faces (the faces need not be regular).

(Of course, you can break these rules in various ways, say, by making square faces. But the units tend to buckle when forced to do this. Or you could make the vertices have degree 4 (I know of two ways you can do this), and that's interesting too. But for this exposition I'll just stick to the above rules.)

The unit itself is very simple, and the first thing I made with it was a dodecahedron. But then I noticed that the locking mechanism in this unit is particularly strong, allowing one to make much larger structures. Thus began my quest to see exactly what other polyhedra you could make with this unit and, later on, how to properly 3-color them.

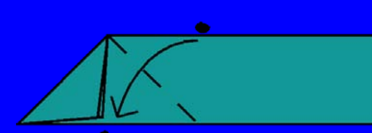
But first let's see how to fold and lock the PHiZZ units together.

How to make the PHiZZ unit

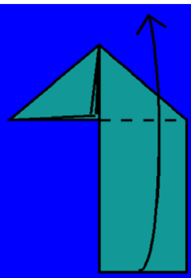


(1) Take a square piece of paper, white side up, and accordion pleat it into fourths.

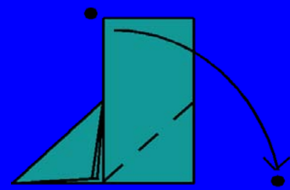
(2) Fold the top left corner down.



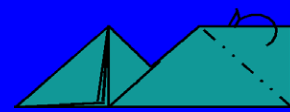
(3) Fold the right end of the strip down to meet the folded edges.



(4) Now fold the strip up, making the bottom flush.

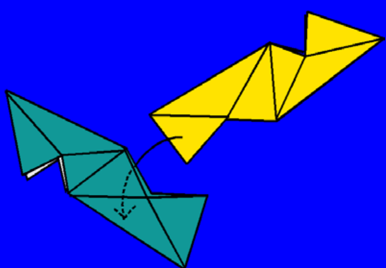


(5) Fold the strip down to the right.

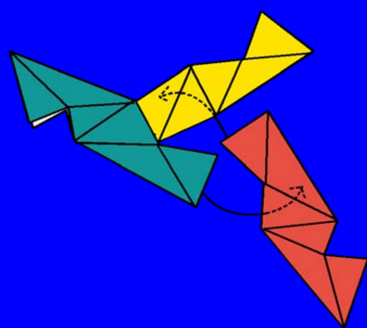


(6) Mountain-fold the upper right corner behind, and you're done with one unit!

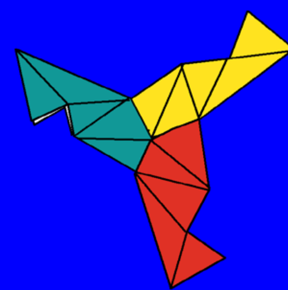
Putting the units together



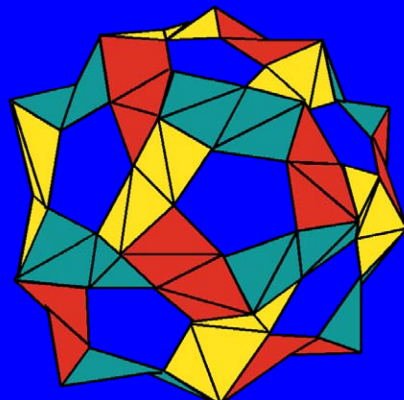
(1) Slide the end of one unit into the side of another, as shown. The flap should go in between the layers of paper, and crease lines should line up.



(2) It takes three units to make a "corner" of the polyhedron. Slide the third unit into the second unit, and slide the first into the third.



(3) This is what you should see. Notice how exactly **half** of each unit was used to make this "pyramid." Continue adding units to this to make your polyhedron.



Adding units to your first three will begin to make a polyhedron, where each unit represents an edge of the polyhedron. Thus, the polyhedra made from this unit will look like polyhedral "frames" where only the edges are visible. The above object is a dodecahedron made from 30 pentagon-hexagon Z-units. It is the smallest structure that can be made from this unit. To make it, keep adding units onto your first three, but when you see five units going around a face, link them up to complete the face. This will make each face a pentagon, and will give you a dodecahedron.

There are many different structures that can be made from this unit. In the next few weeks (or months??) I hope to add more to these pages about how to make bigger things and how many units it takes. I also have been exploring different ways you can color them. Until I write more about these subjects, think about the following exercises:

(1) Make a soccer ball, aka truncated icosahedron, aka Buckminster fullerene, aka C60 molecule from this unit. It requires 90 units.

(2) If 90 units is too scary for you, use 36 units to make a polyhedron with 12 pentagon faces and 2 hexagon faces. Is this polyhedron regular?

(3) Make these objects using only three different colors, so that no two units of the same color touch. This is called a **proper 3-edge coloring** of the polyhedron. Figuring out where to put the colors is a puzzle, but not too hard, for the dodecahedron and the object in (2). But for the soccer ball it is quite a challenge!

Click HERE for more info on making large Buckyballs!

Email: thull@wnec.edu

Back to the Origami Math page