Trapezoidal Rule Example

Use the Trapezoidal rule to approximate $\displaystyle\int_0^{\pi} \sin x \, dx$ using

1. $n = 6$ subintervals,
2. $n = 12$ subintervals, and
3. Richardson extrapolation.

Here $a = 0$, $b = \pi$, and $f(x) = \sin x$.

1. $n = 6 \implies h = \dfrac{b-a}{n} = \dfrac{\pi}{6}$, and $x_i = a + ih = 0 + \frac{\pi}{6} i$, $i = 0, 1, \ldots, 6$.

i	0	1	2	3	4	5	6
x_i	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	$\frac{6\pi}{6}$
f_i	$\sin 0$	$\sin\left(\frac{\pi}{6}\right)$	$\sin\left(\frac{2\pi}{6}\right)$	$\sin\left(\frac{3\pi}{6}\right)$	$\sin\left(\frac{4\pi}{6}\right)$	$\sin\left(\frac{5\pi}{6}\right)$	$\sin\left(\frac{6\pi}{6}\right)$

So

$$\begin{aligned} \int_0^{\pi} \sin x \, dx &\approx \frac{h}{2}\Big[f_0 + 2f_1 + 2f_2 + 2f_3 + 2f_4 + 2f_5 + f_6\Big] \\ &= 1.9540\ 9723\ 3313\,, \qquad \text{Error} = 2.2951\% \end{aligned} \tag{1}$$

2. $n = 12 \implies h = \dfrac{b-a}{n} = \dfrac{\pi}{12}$, and $x_i = a + ih = 0 + \frac{\pi}{12} i$, $i = 0, 1, \ldots, 12$.

i	0	1	2	3	$\cdots$	11	12
x_i	0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\cdots$	$\frac{11\pi}{12}$	$\frac{12\pi}{12}$
f_i	$\sin 0$	$\sin\left(\frac{\pi}{12}\right)$	$\sin\left(\frac{2\pi}{12}\right)$	$\sin\left(\frac{3\pi}{12}\right)$	$\cdots$	$\sin\left(\frac{11\pi}{12}\right)$	$\sin\left(\frac{12\pi}{12}\right)$

So

$$\begin{aligned} \int_0^{\pi} \sin x \, dx &\approx \frac{h}{2}\Big[f_0 + 2f_1 + 2f_2 + 2f_3 + \ldots + 2f_{11} + f_{12}\Big] \\ &= 1.9885\ 6377\ 6584\,, \qquad \text{Error} = 0.5718\% \end{aligned} \tag{2}$$

3. Richardson extrapolation:

$$\begin{aligned} \text{Imp. Est.} &= \text{Better} + \frac{\text{Better} - \text{Poorer}}{2^2 - 1} \\ &= \text{Result}(2) + \frac{\text{Result}(2) - \text{Result}(1)}{2^2 - 1} \\ &= 1.9885\ 6377\ 6584 + \frac{1.9885\ 6377\ 6584 - 1.9540\ 9723\ 3313}{2^2 - 1} \\ &= 2.0000\ 5262\ 4341\,, \qquad \text{Error} = 0.0026\% \end{aligned} \tag{3}$$

NOTE: $\dfrac{\text{Error in Result}(1)}{\text{Error in Result}(2)} = \dfrac{2.2951}{0.5718} = 4.013 \approx 4$ as expected