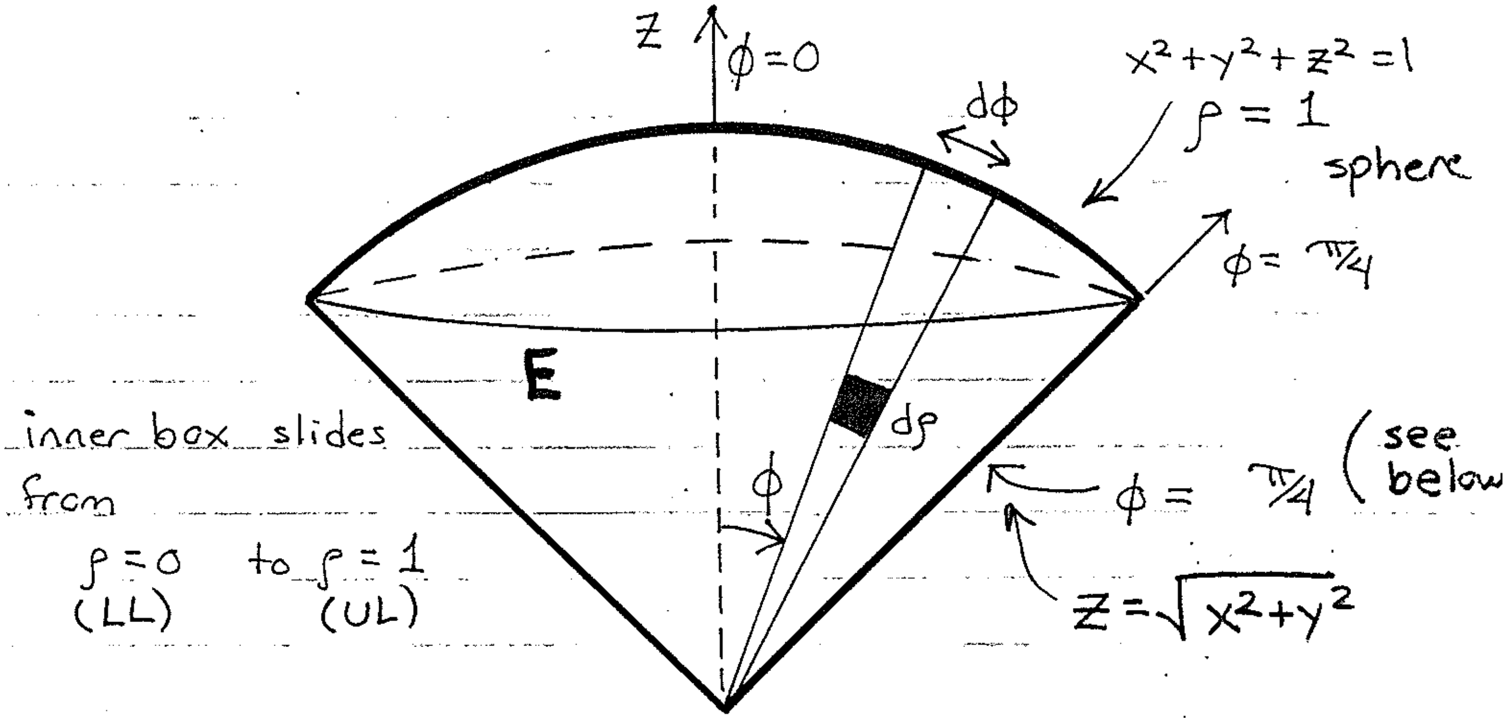
Ex. Find the volume of the "ice cream cone" bound above the cone $z = \sqrt{x^2+y^2}$ and beneath the sphere $x^2+y^2+z^2 = 1$.



Sphere: $x^2+y^2+z^2 = 1 \Rightarrow \rho = 1 \leftarrow$ sphere

Cone: $z = \sqrt{x^2+y^2}$

$$\rho\cos\phi = \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta$$

$$= \rho^2\sin^2\phi\,(\cos^2\theta + \sin^2\theta)$$

$$= \rho^2\sin^2\phi$$

$$\rho\cos\phi = \rho\sin\phi$$

$\Rightarrow$

$\tan\phi = 1 \Rightarrow \phi = \pi/4 \leftarrow$ cone

So inside solid E, ϕ ranges from

$\phi = 0$ (LL) to $\phi = \frac{\pi}{4}$ (UL)

View from above

Middle: $d\theta$

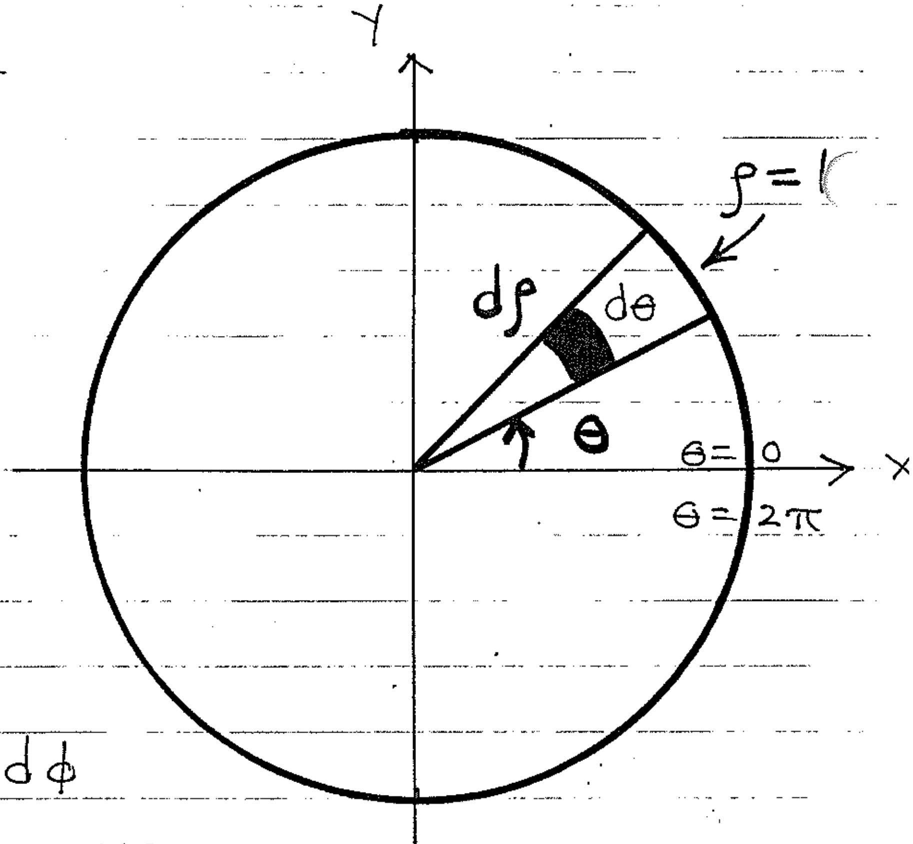
LL: $\theta = 0$

UL: $\theta = 2\pi$

Inner: $d\rho$

LL: $\rho = 0$

UL: $\rho = 1$



$$dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

So

$$V = \iiint_E dV = \iiint_E \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi/4} \int_0^{2\pi} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi/4} \int_0^{2\pi} \sin\phi \cdot \frac{1}{3} \rho^3 \Big|_{\rho=0}^{1} d\theta \, d\phi$$

$$= \frac{1}{3} \int_0^{\pi/4} \int_0^{2\pi} \sin\phi \, d\theta \, d\phi$$

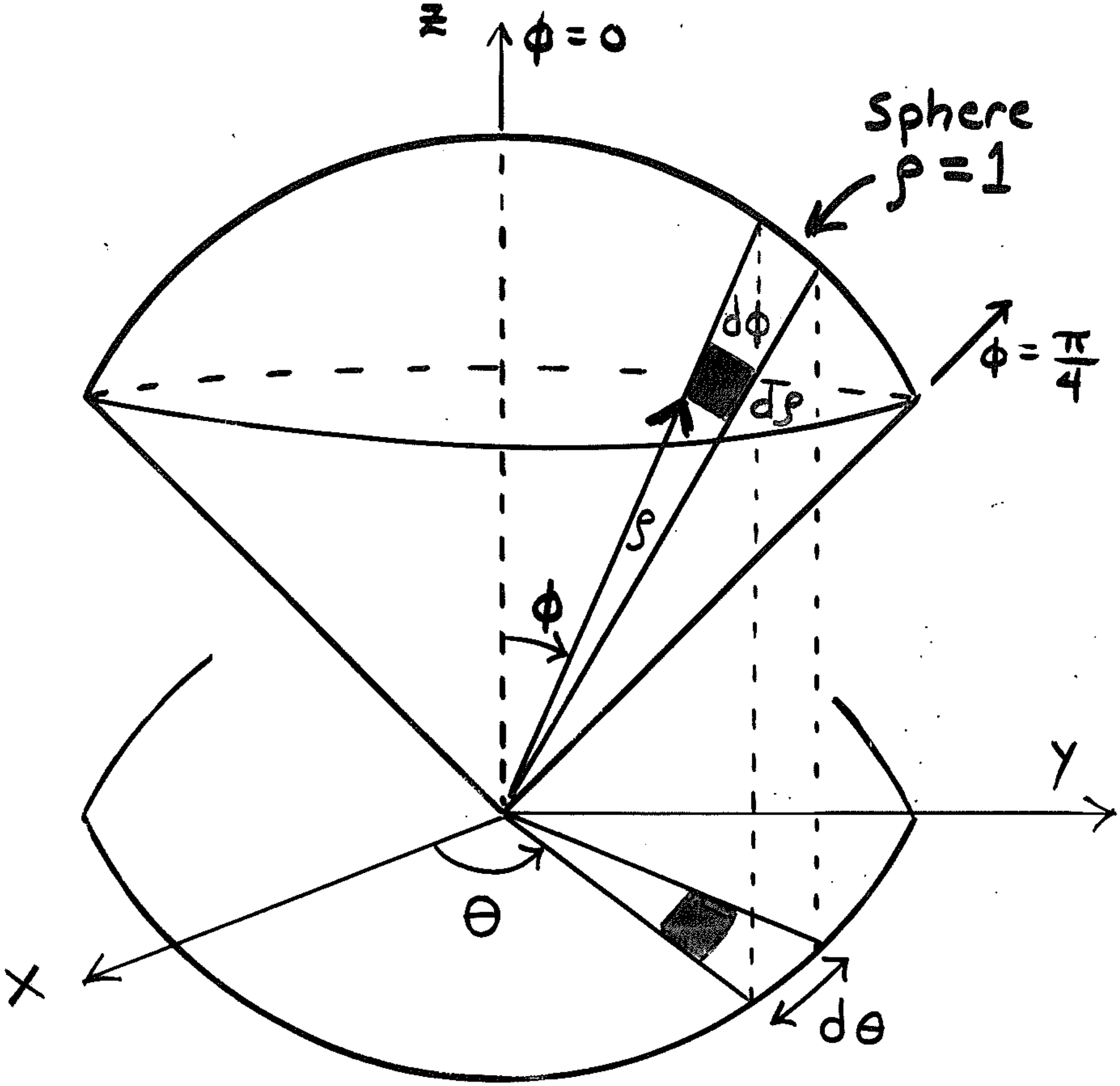
$$= \frac{1}{3} \int_0^{\pi/4} \sin\phi \int_0^{2\pi} d\theta \, d\phi$$

$$= \frac{1}{3} \int_0^{\pi/4} \sin\phi \cdot \theta \Big|_{\theta=0}^{2\pi} d\phi$$

$$= \frac{2}{3}\pi \int_0^{\pi/4} \sin\phi \, d\phi$$

$$= \frac{2}{3}\pi \cdot (-\cos\phi) \Big|_{\phi=0}^{\pi/4}$$

$$= -\frac{2}{3}\pi \left(\cos\frac{\pi}{4} - \cos 0\right) = \frac{2}{3}\pi\left(1 - \frac{\sqrt{2}}{2}\right) = 0.613 \text{ units}^3$$



Outer: $d\phi$

LL: $\phi = 0$ UL: $\phi = \pi/4$

Middle: $d\theta$

LL: $\theta = 0$ UL: $\theta = 2\pi$

Inner: $d\rho$

LL: $\rho = 0$ UL: $\rho = 1$