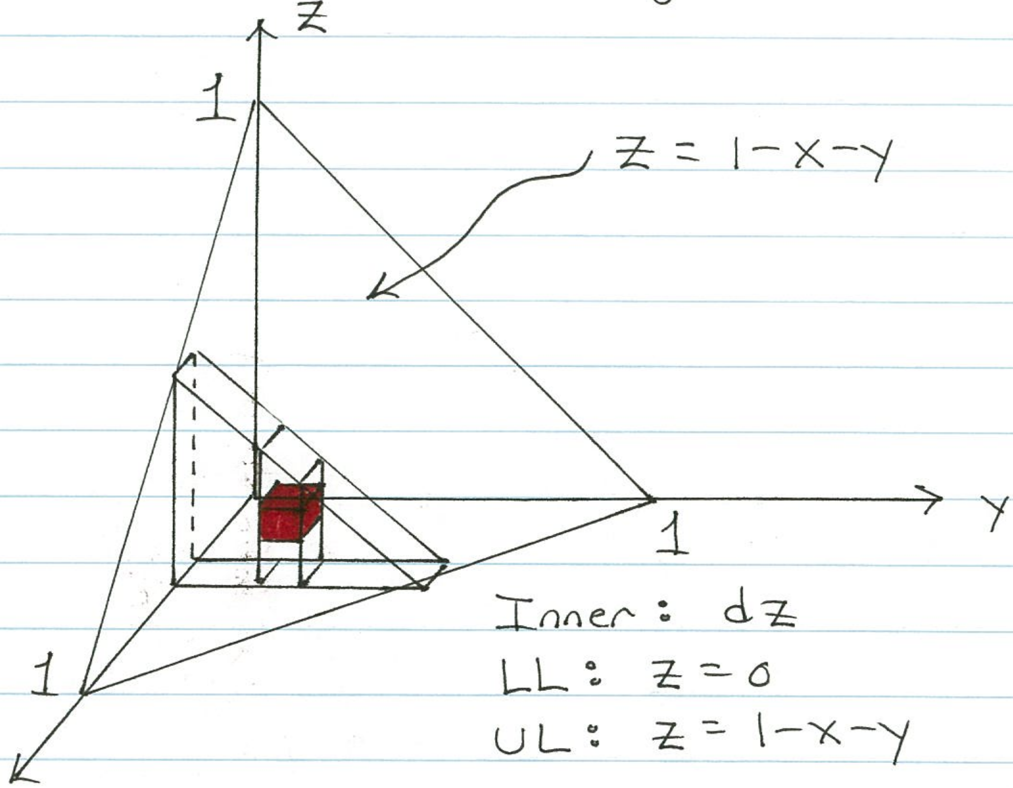
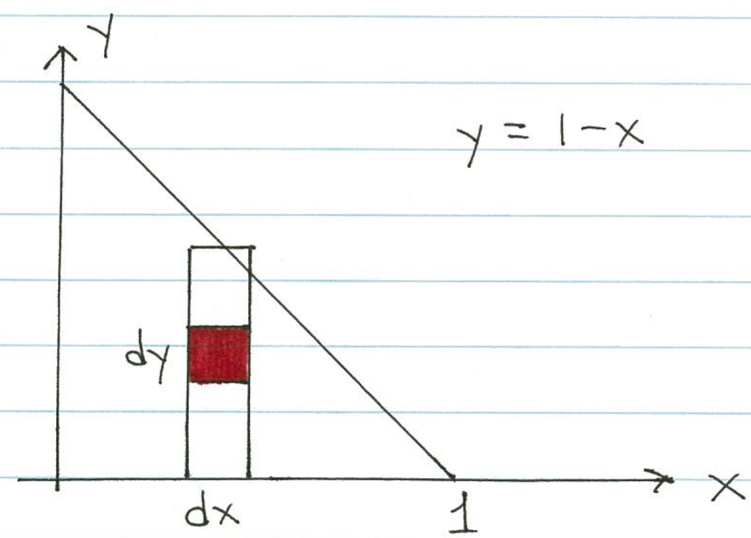
Ex. Find the mass & center of mass of the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$, & $x+y+z=1$ if its density is $\rho(x,y,z) = y$. (gm)

x, y, z measured in cm.



Inner: dz
LL: $z=0$
UL: $z=1-x-y$

Top View:



middle: dy
LL: $y=0$
UL: $y=1-x$

Outer: dx
LL: $x=0$
UL: $x=1$

So $\quad dV = dz\,dy\,dx$

$$dm = \rho\, dV = y\, dV = y\, dz\, dy\, dz$$

$$m = \iiint_E dm = \iiint_E y\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} y\, z \Big|_{z=0}^{1-x-y} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} y(1-x-y)\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left[(1-x)y - y^2\right] dy\, dx$$

$$= \int_0^1 \left[(1-x)\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\right]\Big|_{y=0}^{1-x} dx$$

$$= \frac{1}{6}\int_0^1 \left[3(1-x)y^2 - 2y^3\right]\Big|_{y=0}^{1-x} dx$$

$$= \frac{1}{6}\int_0^1 \left[3(1-x)^3 - 2(1-x)^3\right] dx$$

$$= \frac{1}{6}\int_0^1 (1-x)^3\, dx$$

$u = 1-x \qquad dx = -du$

$LL: u=1 \qquad UL: u=0$

$$= \frac{1}{6}\int_{u=1}^{0} u^3(-du)$$

switch limits & change sign

$$= \frac{1}{6}\int_0^1 u^3\, du$$

$$= \frac{1}{6}\cdot\frac{1}{4}u^4\Big|_{u=0}^{1}$$

$$= \frac{1}{6}\cdot\frac{1}{4} = \frac{1}{24}\text{ gm} \qquad = m$$

Moment about the yz-plane:

$$M_{yz} = \iiint_E dM_{yz} = \iiint_E x\,dm = \iiint_E x\,\rho\,dV$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} xy\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} xyz \Big|_{z=0}^{1-x-y} dy\,dx$$

$$= \int_0^1 \int_0^{1-x} xy(1-x-y)\,dy\,dx$$

$$= \int_0^1 x \int_0^{1-x} \left[(1-x)y - y^2\right] dy\,dx$$

$$= \int_0^1 x \left[(1-x)\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\right]\Big|_{y=0}^{1-x} dx$$

$$= \frac{1}{6}\int_0^1 x\left[3(1-x)y^2 - 2y^3\right]\Big|_{y=0}^{1-x} dx$$

$$= \frac{1}{6}\int_0^1 x\left[3(1-x)^3 - 2(1-x)^3\right] dx$$

$$= \frac{1}{6}\int_0^1 x(1-x)^3\,dx$$

$u = 1-x \quad (x = 1-u)$

$du = -dx$

LL: $u=1$ UL: $x=0$

sub

$$= \frac{1}{6}\int_{u=1}^{0} (1-u)u^3(-du)$$

switch limits & change sign

$$= \frac{1}{6}\int_0^1 (u^3 - u^4)\,du$$

$$= \frac{1}{6}\left[\tfrac{1}{4}u^4 - \tfrac{1}{5}u^5\right]\Big|_{u=0}^{1}$$

$$= \frac{1}{6}\left(\frac{1}{4} - \frac{1}{5}\right)$$

$$= \frac{1}{6}\cdot\frac{1}{20} = \frac{1}{120}\ \text{gm}\cdot\text{cm} = M_{yz}$$

Moment about the xz-plane:

$$M_{xz} = \iiint_E dM_{xz} = \iiint_E y\,dm = \iiint_E y\,\rho\,dV$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y^2\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} y^2 z \Big|_{z=0}^{1-x-y} dy\,dx$$

$$= \int_0^1 \int_0^{1-x} y^2 (1-x-y)\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[(1-x)y^2 - y^3\right] dy\,dx$$

$$= \int_0^1 \left[(1-x)\tfrac{1}{3}y^3 - \tfrac{1}{4}y^4\right]\Big|_{y=0}^{1-x} dx$$

$$= \frac{1}{12}\int_0^1 \left[4(1-x)y^3 - 3y^4\right]\Big|_{y=0}^{1-x} dx$$

$$= \frac{1}{12}\int_0^1 \left[4(1-x)^4 - 3(1-x)^3\right] dx$$

$u = 1-x \quad du = -dx \quad dx = -du$

LL: $u = 1 \quad$ UL: $u = 0$

$$= \frac{1}{12}\int_1^0 (4u^4 - 3u^3)(-du)$$

switch limits & change sign

$$= \frac{1}{2}\left(\frac{4}{5}u^5 - \frac{3}{4}u^4\right)\Big|_{u=0}^{1}$$

$$= \frac{1}{2}\left(\frac{4}{5} - \frac{3}{4}\right)$$

$$= \frac{1}{2}\left(\frac{1}{20}\right) = \frac{1}{40}\ \text{gm}\cdot\text{cm} = M_{xz}$$

Moment about the xy-plane:

$$M_{xy} = \iiint_E dM_{xy} = \iiint_E z\,dm = \iiint_E z\rho\,dV$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} zy\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2} y z^2 \Big|_{z=0}^{1-x-y} dy\,dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} y(1-x-y)^2\,dy\,dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} y\left[(1-x)^2 - 2(1-x)y + y^2\right] dy\,dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} \left[(1-x)^2 y - 2(1-x)y^2 + y^3\right] dy\,dx$$

$$= \frac{1}{2} \int_0^1 \left[(1-x)^2 \frac{1}{2} y^2 - 2(1-x)\frac{1}{3}y^3 + \frac{1}{4}y^4\right] \Big|_{y=0}^{1-x} dx$$

$$= \frac{1}{2} \cdot \frac{1}{12} \int_0^1 \left[6(1-x)^2 y^2 - 8(1-x)y^3 + 3y^4\right] \Big|_{y=0}^{1-x} dx$$

$$= \frac{1}{24} \int_0^1 \left[6(1-x)^4 - 8(1-x)^4 + 3(1-x)^4\right] dx$$

$$= \frac{1}{24} \int_0^1 (1-x)^4\,dx \qquad u = 1-x \quad du = -dx \quad \text{LL: } u = 1 \quad \text{UL: } u = 0$$

$$= \frac{1}{24} \int_1^0 u^4(-du) = \frac{1}{24} \cdot \frac{1}{5} u^5 \Big|_{u=0}^{1} = \frac{1}{120} \text{ gm}\cdot\text{cm}$$

So

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1/120 \text{ gm}\cdot\text{cm}}{1/24 \text{ gm}} = \frac{1}{5} \text{ cm}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1/40 \text{ gm}\cdot\text{cm}}{1/24 \text{ gm}} = \frac{3}{5} \text{ cm}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1/120 \text{ gm}\cdot\text{cm}}{1/24 \text{ gm}} = \frac{1}{5} \text{ cm}$$

Center of mass: $\left(\frac{1}{5}, \frac{3}{5}, \frac{1}{5}\right)$