10.4 Areas in Polar Coordinates

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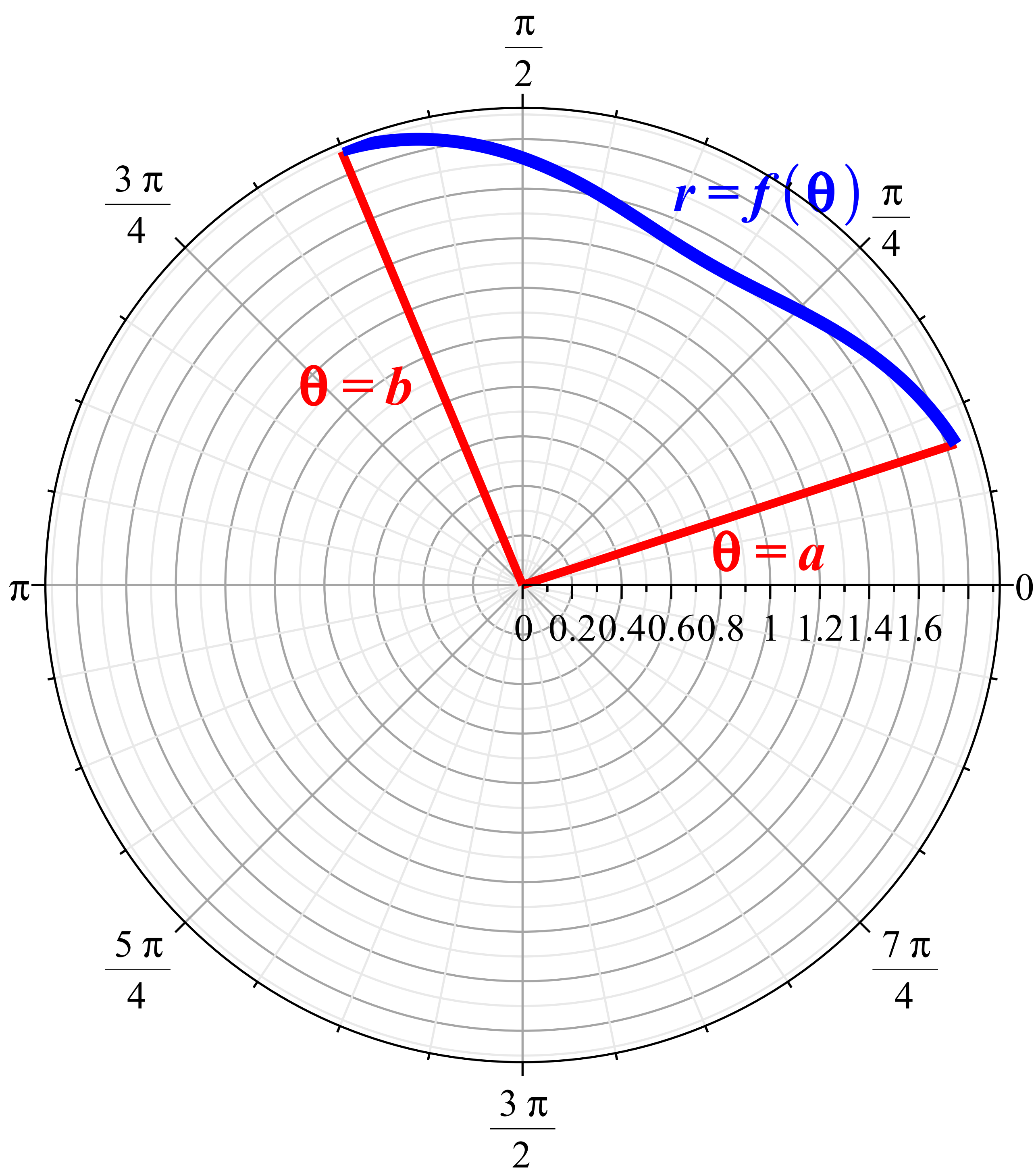
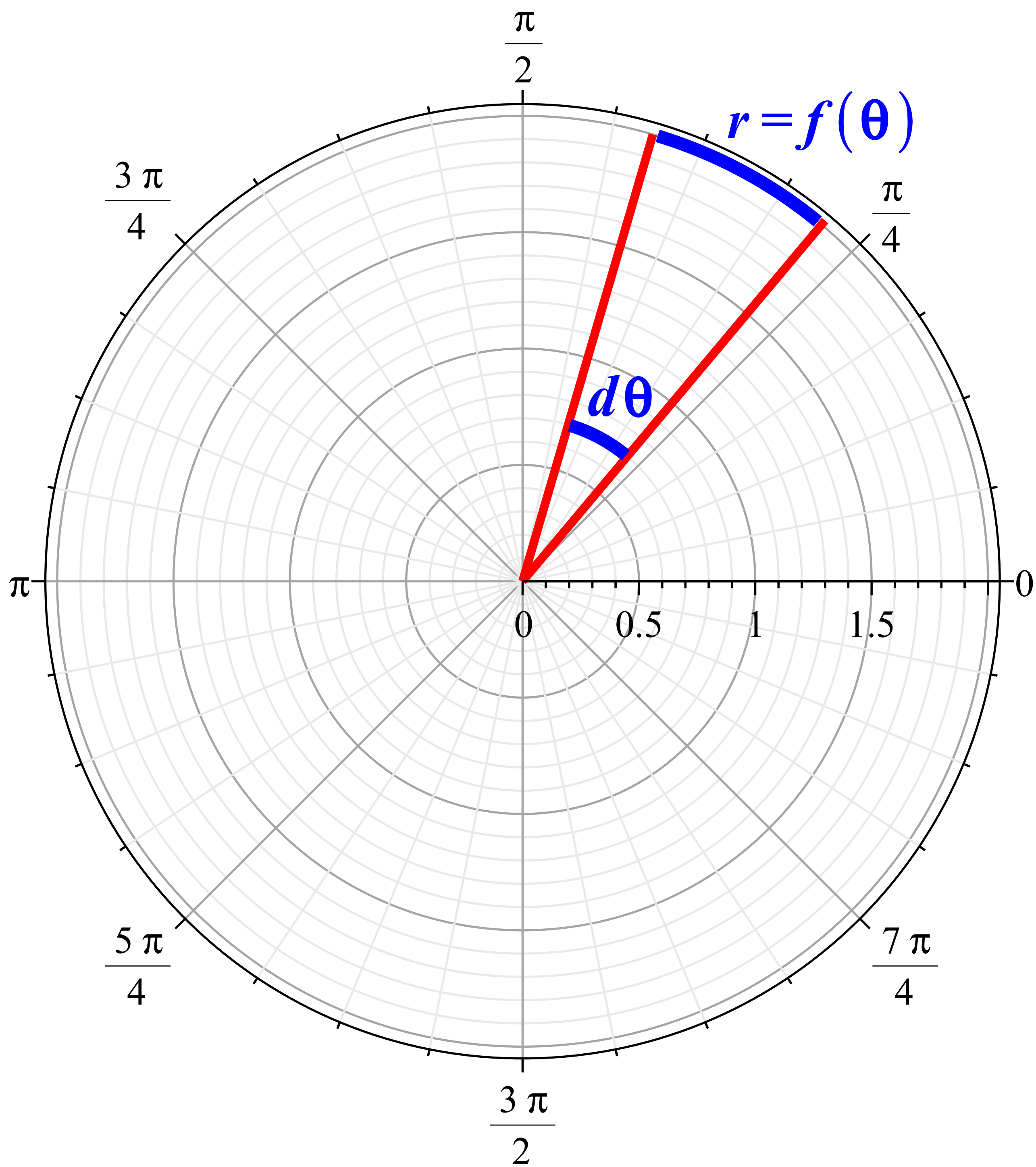
Determine the area of the region bound by $r = f(\theta)$ and the rays $\theta = a$ and $\theta = b$:





Now consider the infinitesimal sector inside the region traversed by an infinitesimal change $d\theta$ in the angle. Since the angle $d\theta$ is infinitesimal, the sector has infinitesimal area dA given by

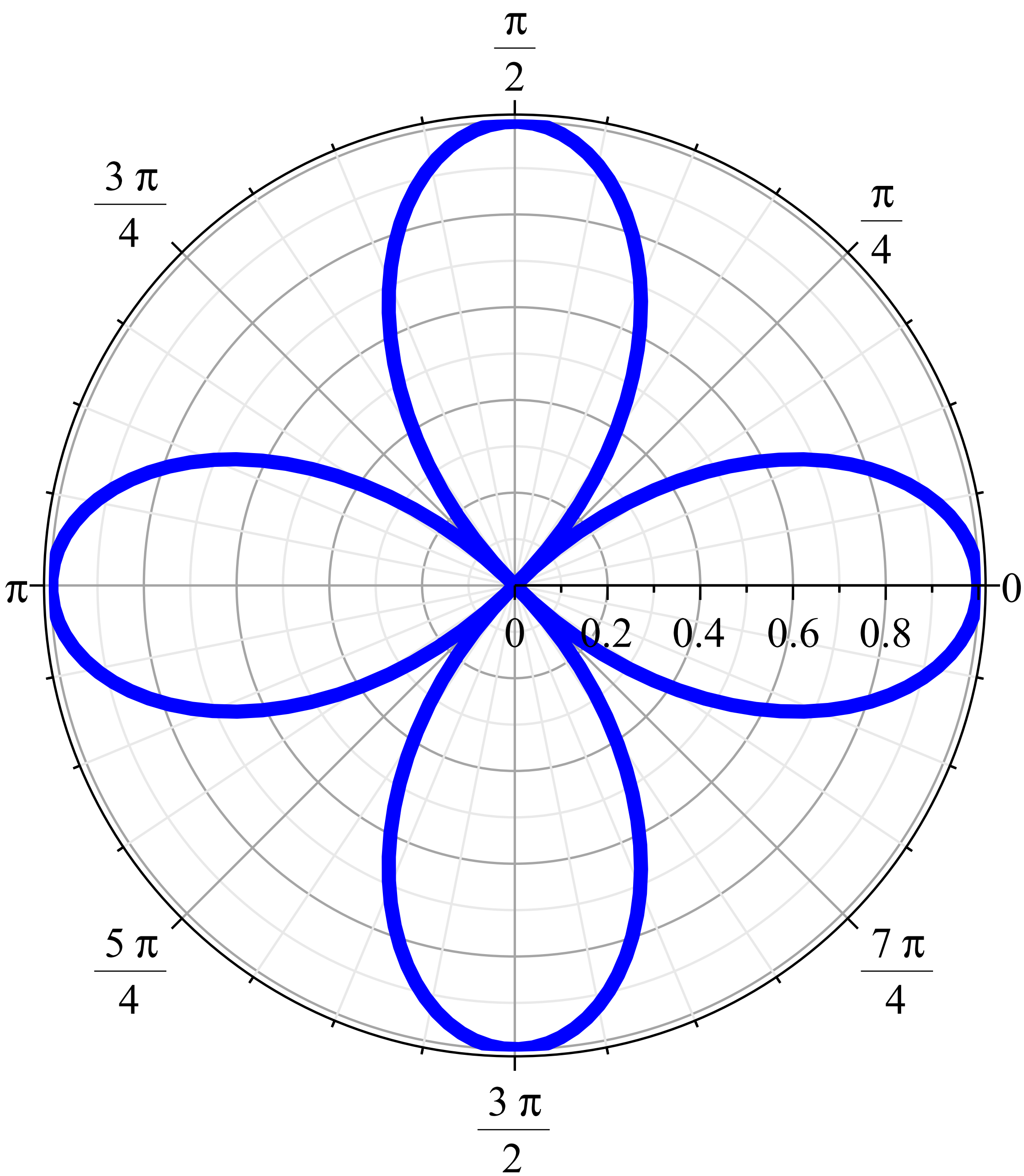
$$dA = \frac{1}{2} r^2 d\theta .$$

Therefore the region bound by $r = f(\theta)$ and the rays $\theta = a$ and $\theta = b$ has area

$$A = \int dA = \int_{\theta=a}^{b} \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_{\theta=a}^{b} r^2 d\theta .$$

Example 1: Determine the area of one petal of the rose given by $r = \cos 2\theta$. Since $n = 2$ (even), the rose as 4 petals on $0 \leq \theta \leq 2\pi$.

```
> restart ;
> with(plots):  with(plottools):
> polarplot(cos(2*theta),axesfont=[times,roman,14],thickness=6,
  color=blue);
```



The top half of the right-most petal is spanned by letting θ vary from $\theta = 0$ to $\theta = \frac{\pi}{4}$, so the area of one petal is

$$A = 2 \cdot \frac{1}{2} \int_{\theta=0}^{\frac{\pi}{4}} r^2 \, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{4}} \frac{1}{2}\left(1+\cos 4\theta\right)\, d\theta, \qquad \text{trig identity:} \quad \cos^2 x = \frac{1}{2}\left(1+\cos 2x\right)$$

$$= \frac{1}{2}\left(\theta + \frac{1}{4}\sin 4\theta\right)\Bigg|_{0}^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{4} + \frac{1}{4}\sin\pi\right) - \left(0 + \frac{1}{4}\sin 0\right)\right]$$

$$= \frac{\pi}{8}\ \text{units}^2$$

Check:

```
> r := cos(2*theta);
```

$$r := \cos(2\,\theta) \qquad \textbf{(1)}$$

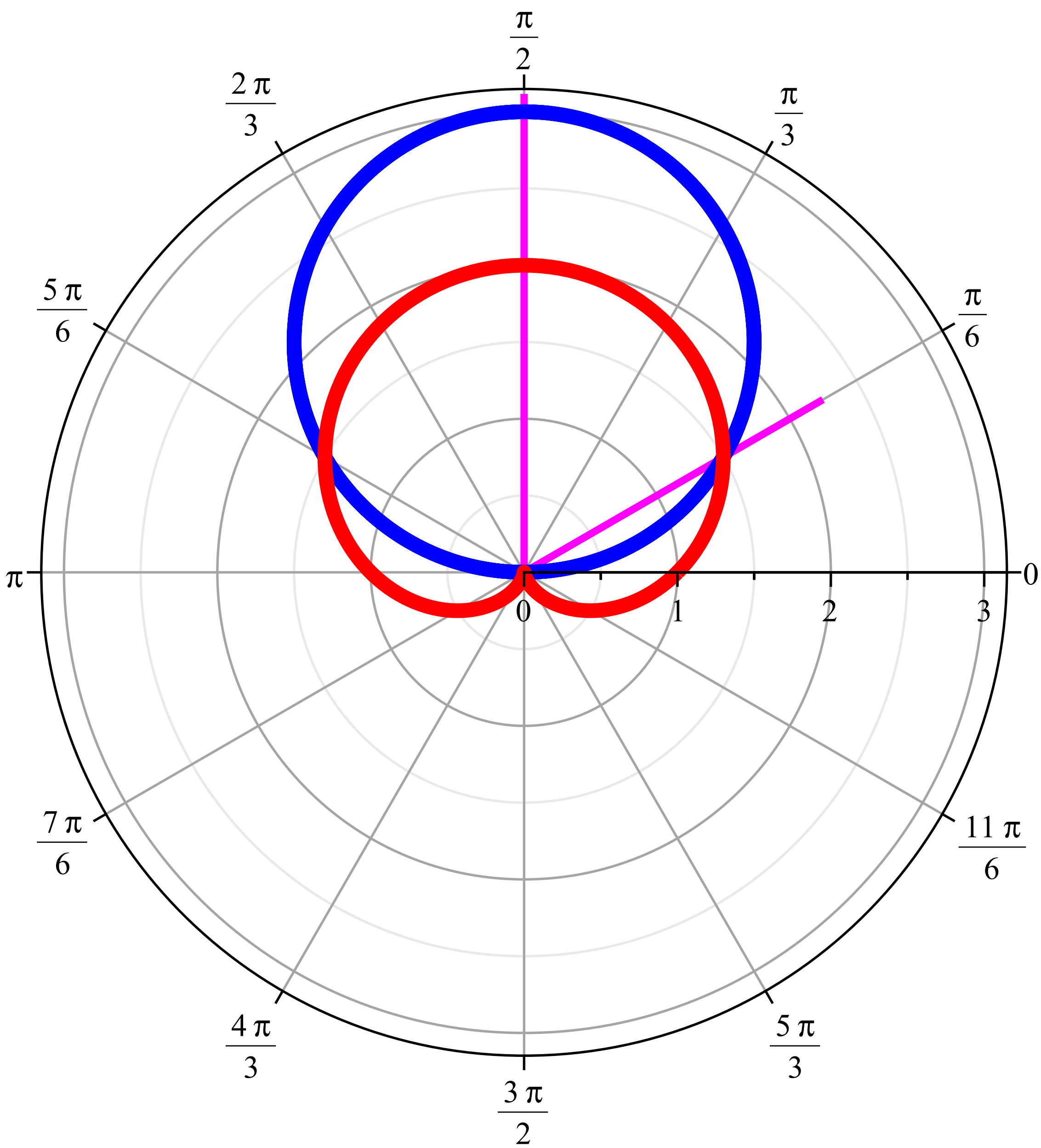
```
> Area := 2 * 1/2 * int( r^2, theta = 0 .. Pi/4 );
```

$$Area := \frac{1}{8}\pi \qquad \textbf{(2)}$$

Example 2: Determine the area of the region inside the circle $r_o = 3\sin\theta$ and outside the cardioid $r_i = 1+\sin\theta$.

The circle has radius $\frac{3}{2}$ and center $\left(0, \frac{3}{2}\right)$.

```
> circ := theta -> 3*sin(theta):
> ri := theta -> 1+sin(theta):
> ycircle:=polarplot(circ(theta),thickness=6, color=blue):
> ray1:=line([0, 0], [1.5*circ(Pi/6)*cos(Pi/6),1.5*circ(Pi/6)*sin
  (Pi/6)], color = magenta, linestyle = solid, thickness=3):
> ray2:=line([0, 0], [1.2*circ(Pi/2)*cos(Pi/2),1.2*circ(Pi/2)*sin
  (Pi/2)], color = magenta, linestyle = solid, thickness=3):
> cardioid:=polarplot(ri(theta),thickness=6, color=red):
> display([ray1,ray2, ycircle,cardioid],coordinateview=[-1..3.1,0.
  .2*Pi]);
```



Let's determine the area of the right half and multiply by 2. So we must determine at what value of θ the curves intersect in Quadrant I:

Set $r = 3\sin\theta \equiv 1 + \sin\theta$ and solve for θ:

$$\Rightarrow \quad 2\sin\theta \equiv 1$$

$$\Rightarrow \quad \sin\theta \equiv \frac{1}{2}$$

$$\Rightarrow \quad \theta \equiv \frac{\pi}{6}.$$

So

$$A = 2 \cdot \frac{1}{2} \int_{\theta=\frac{\pi}{6}}^{\frac{\pi}{2}} \left(r_o^2 - r_i^2\right) d\theta$$

$$= \int_{\theta=\frac{\pi}{6}}^{\frac{\pi}{2}} \left[(3 \sin\theta)^2 - (1 + \sin\theta)^2 \right] d\theta$$

$$= \int_{\theta=\frac{\pi}{6}}^{\frac{\pi}{2}} \left[9 \sin^2\theta - 1 - 2 \sin\theta - \sin^2\theta \right] d\theta$$

$$= \int_{\theta=\frac{\pi}{6}}^{\frac{\pi}{2}} \left[8 \sin^2\theta - 1 - 2 \sin\theta \right] d\theta$$

$$= \int_{\theta=\frac{\pi}{6}}^{\frac{\pi}{2}} \left[8 \cdot \frac{1}{2} (1 - \cos 2\theta) - 1 - 2 \sin\theta \right] d\theta \quad \text{trig identity: } \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$= \int_{\theta=\frac{\pi}{6}}^{\frac{\pi}{2}} \left[3 - 4 \cos 2\theta - 2 \sin\theta \right] d\theta$$

$$= \left[3\theta - \frac{4}{2} \sin 2\theta + 2 \cos\theta \right] \Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 3\left(\frac{\pi}{2} - \frac{\pi}{6} \right) - 2\left(\sin\pi - \sin\frac{\pi}{3} \right) + 2\left(\cos\frac{\pi}{2} - \cos\frac{\pi}{6} \right)$$

$$= 3\left(\frac{\pi}{3} \right) - 2\left(0 - \frac{\sqrt{3}}{2} \right) + 2\left(0 - \frac{\sqrt{3}}{2} \right)$$

$$= \pi + \sqrt{3} - \sqrt{3}$$

$$= \pi \; units^2 .$$

Check:

```
> ro := 3 * sin(theta);    ri := 1 + sin(theta);
```

$$ro := 3\sin(\theta)$$

$$ri := 1 + \sin(\theta) \tag{3}$$

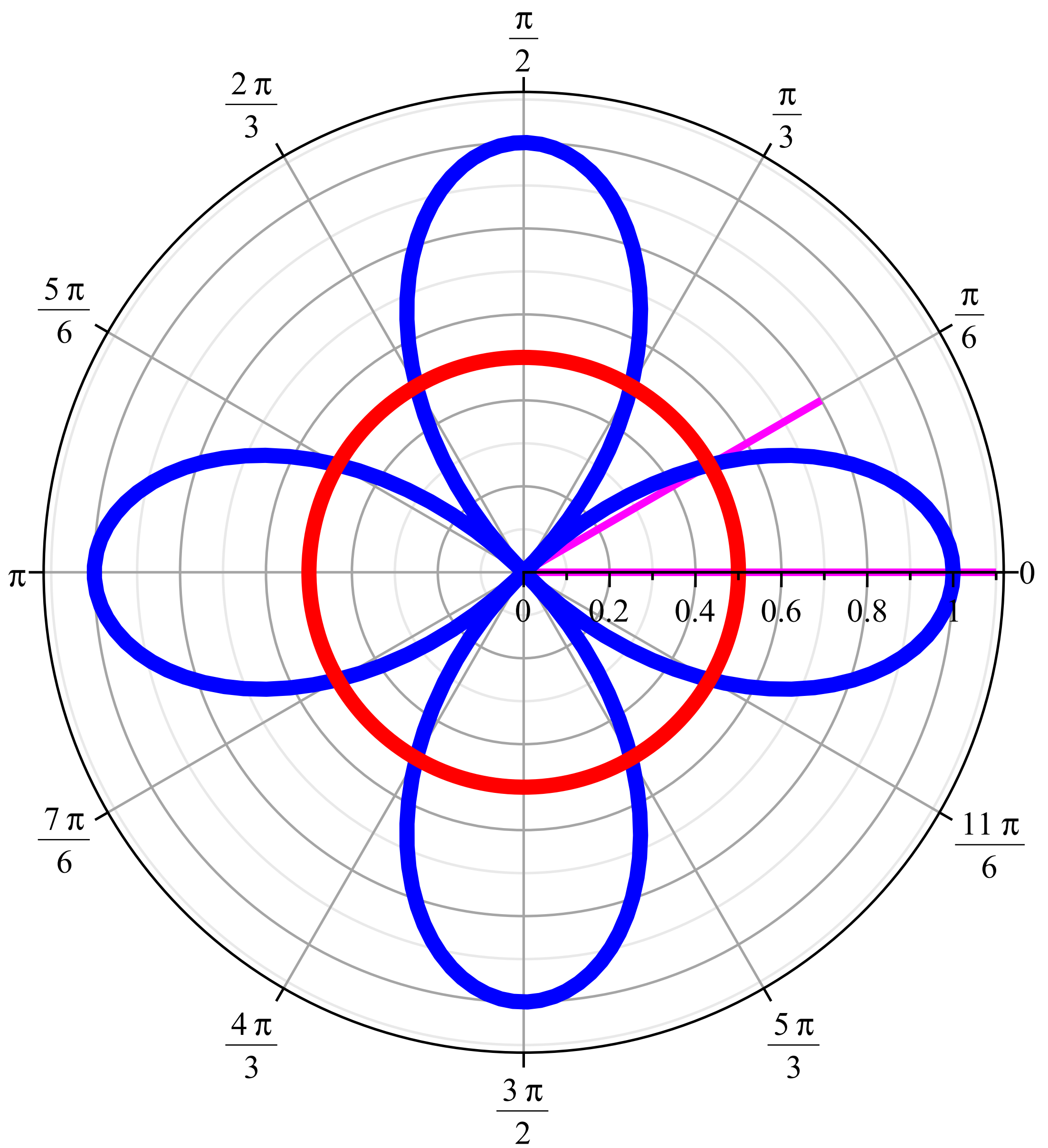
```
> Area := 2 * 1/2 * int( ro^2 - ri^2, theta = Pi/6 .. Pi/2);
```

$$Area := \pi \tag{4}$$

Example 3: Determine the area of the region inside the rose $r_o = \cos 2\theta$ and outside the circle $r_i = \frac{1}{2}$.

Since $n = 2$ (even), the rose as 4 petals on $0 \leq \theta \leq 2\pi$.

```
> ro := theta -> cos(2*theta):
> ri := theta -> 1/2:
> rose:=polarplot(ro(theta),thickness=6, color=blue):
> circ:=polarplot(ri(theta),thickness=6, color=red):
> ray1:=line([0, 0], [1.1*ro(0)*cos(0),1.1*ro(0)*sin(0)], color =
  magenta, linestyle = solid, thickness=3):
> ray2:=line([0, 0], [1.6*ri(Pi/6)*cos(Pi/6),1.6*ri(Pi/6)*sin(Pi/6)
  ], color = magenta, linestyle = solid, thickness=3):
> display([ray1,ray2, rose,circ],coordinateview=[-1..1.1,0..2*Pi]);
```



Let's determine the area of the upper half inside the right-most petal multiply by 8. So we must determine at what value of θ the curves intersect in Quadrant I:

Set $\quad r = \cos 2\theta \equiv \frac{1}{2}$ and solve for θ:

$$\Rightarrow \quad 2\theta = \frac{\pi}{3}$$

$$\Rightarrow \quad \theta = \frac{\pi}{6}.$$

So

$$A = 8 \cdot$$

$$\frac{1}{2}\int_{\theta=0}^{\frac{\pi}{6}} \left(r_o^2 - r_i^2\right)\, d\theta$$

$$= \quad 4\int_{\theta=0}^{\frac{\pi}{6}} \left[\cos^2 2\theta - \frac{1}{4}\right] d\theta$$

$$= \quad 4\int_{\theta=0}^{\frac{\pi}{6}} \left[\cos^2 2\theta - \frac{1}{4}\right] d\theta \qquad \text{trig identity:} \quad \cos^2 x = \frac{1}{2}\left(1 + \cos 2x\right)$$

$$= \quad 4\int_{\theta=0}^{\frac{\pi}{6}} \left[\frac{1}{2}\left(1 + \cos 4\theta\right) - \frac{1}{4}\right] d\theta$$

$$= \quad \int_{\theta=0}^{\frac{\pi}{6}} \left[1 + 2\cos 4\theta\right] d\theta$$

$$= \quad \left[\theta + \frac{2}{4}\sin 4\theta\right] \Bigg|_{0}^{\frac{\pi}{6}}$$

$$= \quad \left(\frac{\pi}{6} + \frac{1}{2}\sin\frac{2\pi}{3}\right) - \left(0 + \sin 0\right)$$

$$= \quad \frac{\pi}{6} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \quad \frac{\pi}{6} + \frac{\sqrt{3}}{4} \quad units^2 .$$

Check:

```
> ro := cos(2*theta);     ri := 1/2;
```

$$ro := \cos\left(2\,\theta\right)$$

$$ri := \frac{1}{2} \tag{5}$$

```
> Area := 8 * 1/2 * int( ro^2 - ri^2, theta = 0 .. Pi/6 );
```

$$Area := \frac{1}{4}\sqrt{3} + \frac{1}{6}\pi \tag{6}$$

```
>
```