10.3 Polar Coordinates

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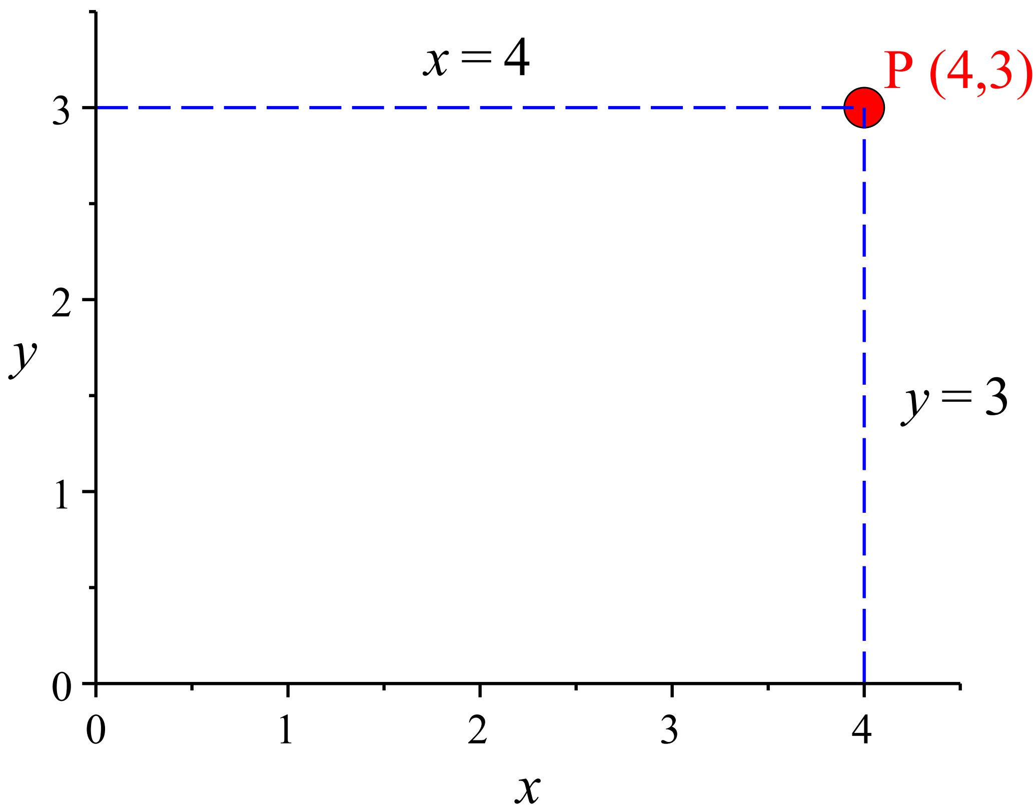
With Cartesian coordinates (x, y) of a point P,

a) x denotes the distance from the y axis to P, and
b) y denotes the distance from the x axis to P.

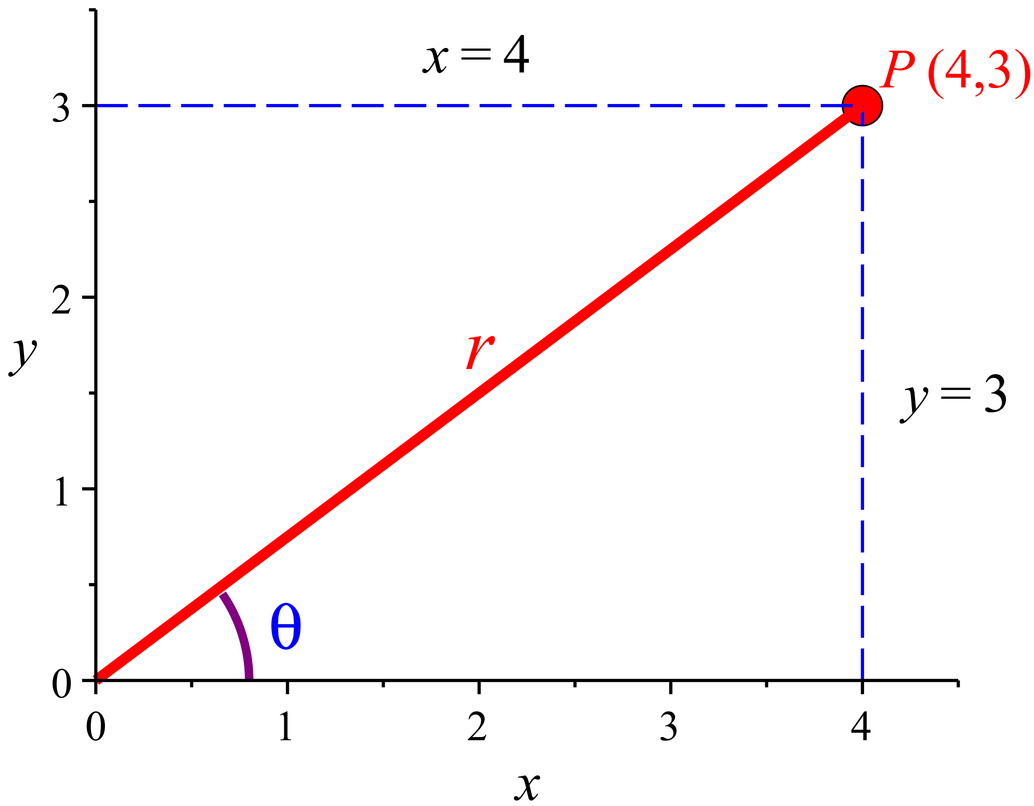


We can define an alternative coordinate system , called **polar coordinates**, in the following manner.

1) Define the ray $\underline{OP}$ from the origin O to P.
2) Let r denote the length of ray $\underline{OP}$.
3) Let θ denote the angle from the $+x$ axis to ray $\underline{OP}$.
   The $+x$ axis is called the **polar axis**.

Angles measured counterclockwise from the polar axis are called positive.
Angles measured clockwise from the polar axis are called negative.

4) Then the polar coordinates of point P are (r, θ).



In the above example, $r = \sqrt{x^2+y^2} = \sqrt{(4)^2+(3)^2} = 5$.

The angle θ is determined by

$$\tan\theta = \frac{y}{x} = \frac{3}{4} \Rightarrow \theta = \tan^{-1}\left(\frac{3}{4}\right) = 0.6425 \text{ rad} = 36.87^{\circ}.$$

So point P has polar coordinates $(r, \theta) = (5, 0.6425) = (5, 36.87^{\circ})$.

In general then, if a point P has Cartesian coordinates (x, y), then its polar coordinates (r, θ) are determined by the **transformation equations**:

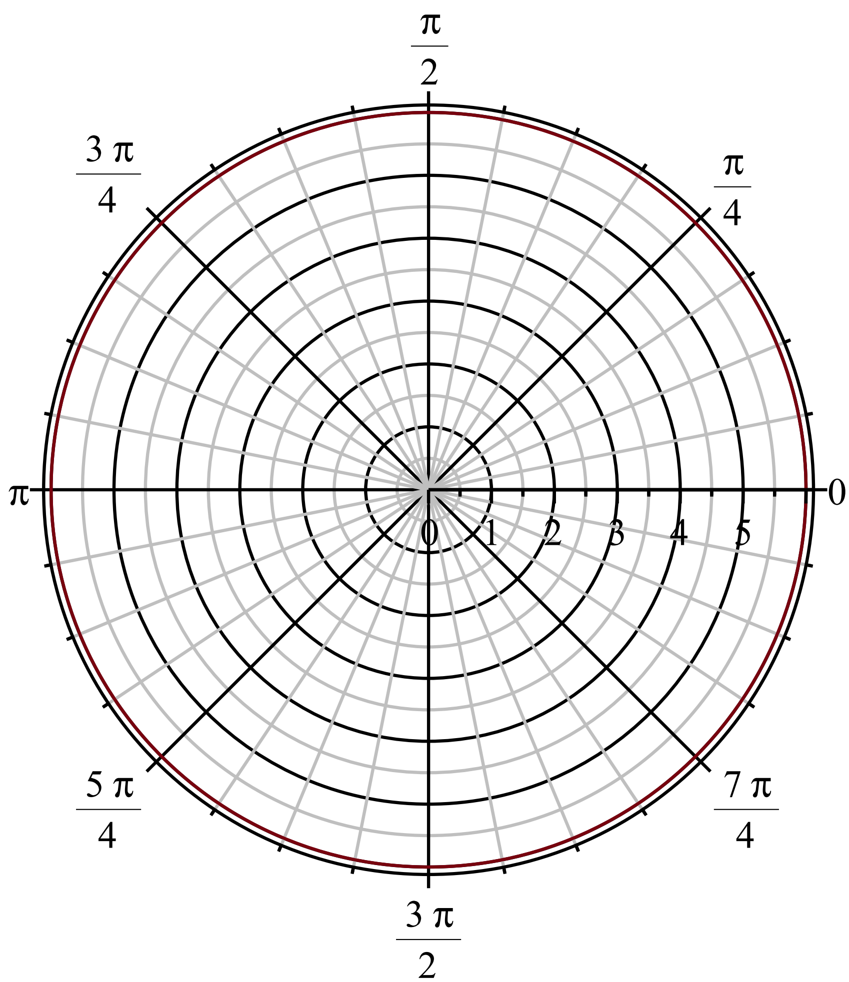
$$r^2 = x^2 + y^2, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$ (use these to convert Cartesian coords to polar coords)

$x = r\cos\theta\,, \qquad y = r\sin\theta\,.$ (use these to convert polar coords to Cartesian coords)

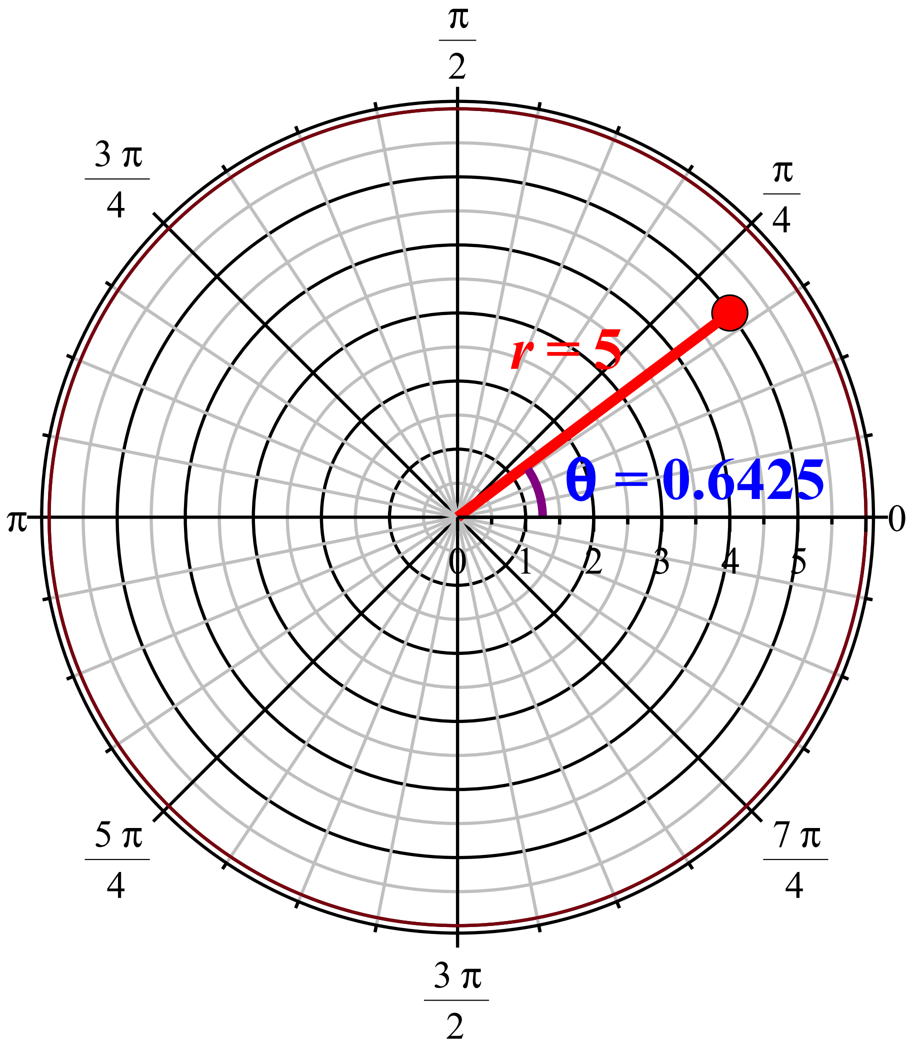
We define the origin in polar coordinates, called the **pole**, by

$$(r, \theta) = (0, \theta) \quad \text{for any angle } \theta.$$

Good polar graph paper would look like



Here's the point $(x, y) = (4, 3)$, *i.e.* $(r, \theta) = (5, 0.6425)$ plotted in polar coordinates:



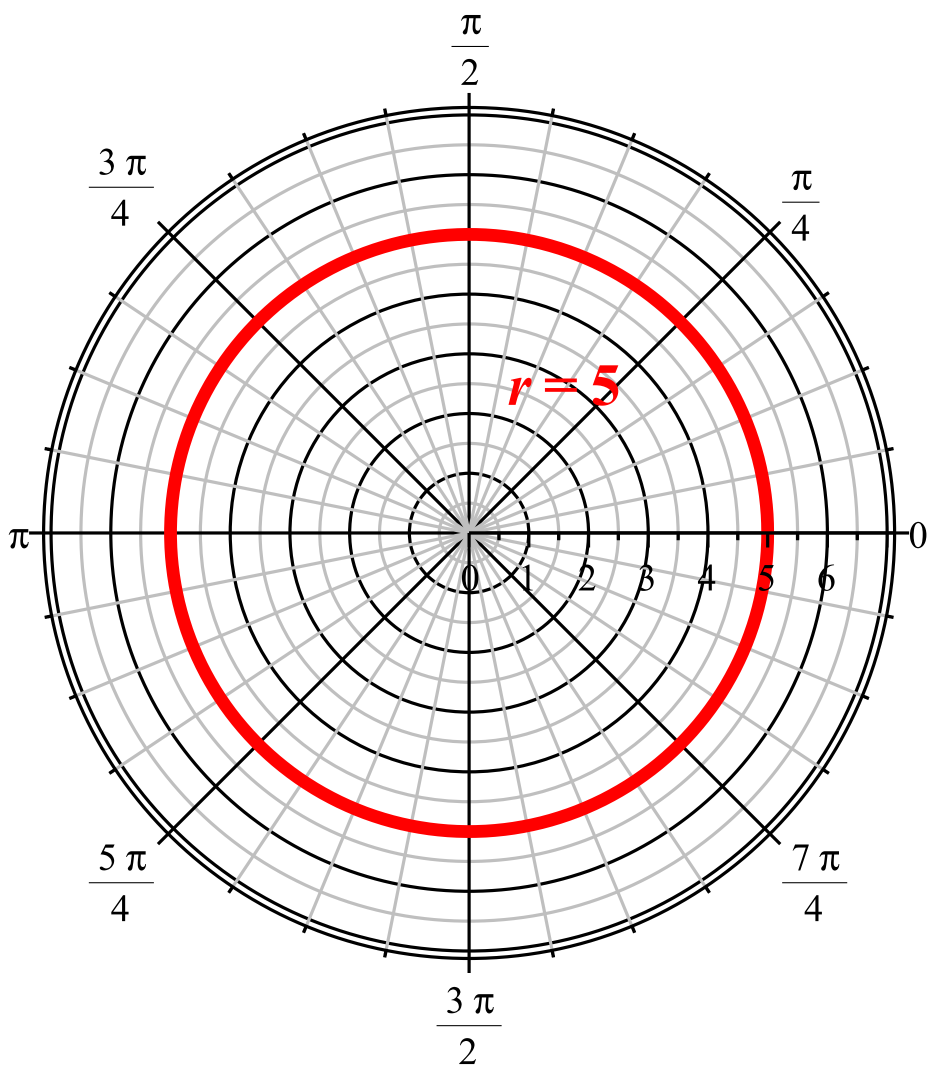
Note: We've merely described the point P in space using different coordinate systems.

Curves in Polar Coordinates

A curve in polar coordinates is usually represented by a formula of the form

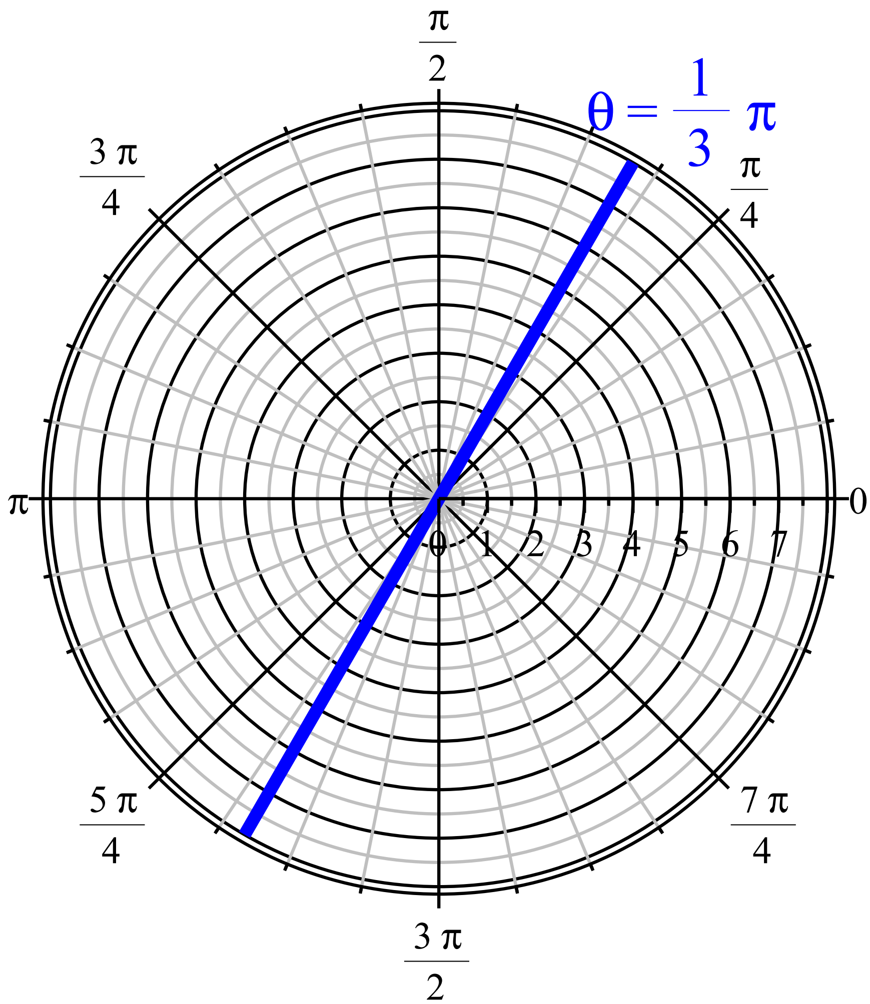
$$r = f(\theta).$$

Example: The simple expression $r = 5$ is independent of θ, so it represents the set of all points that are 5 units from the origin, *i.e.*, a circle of radius 5 centered at O.

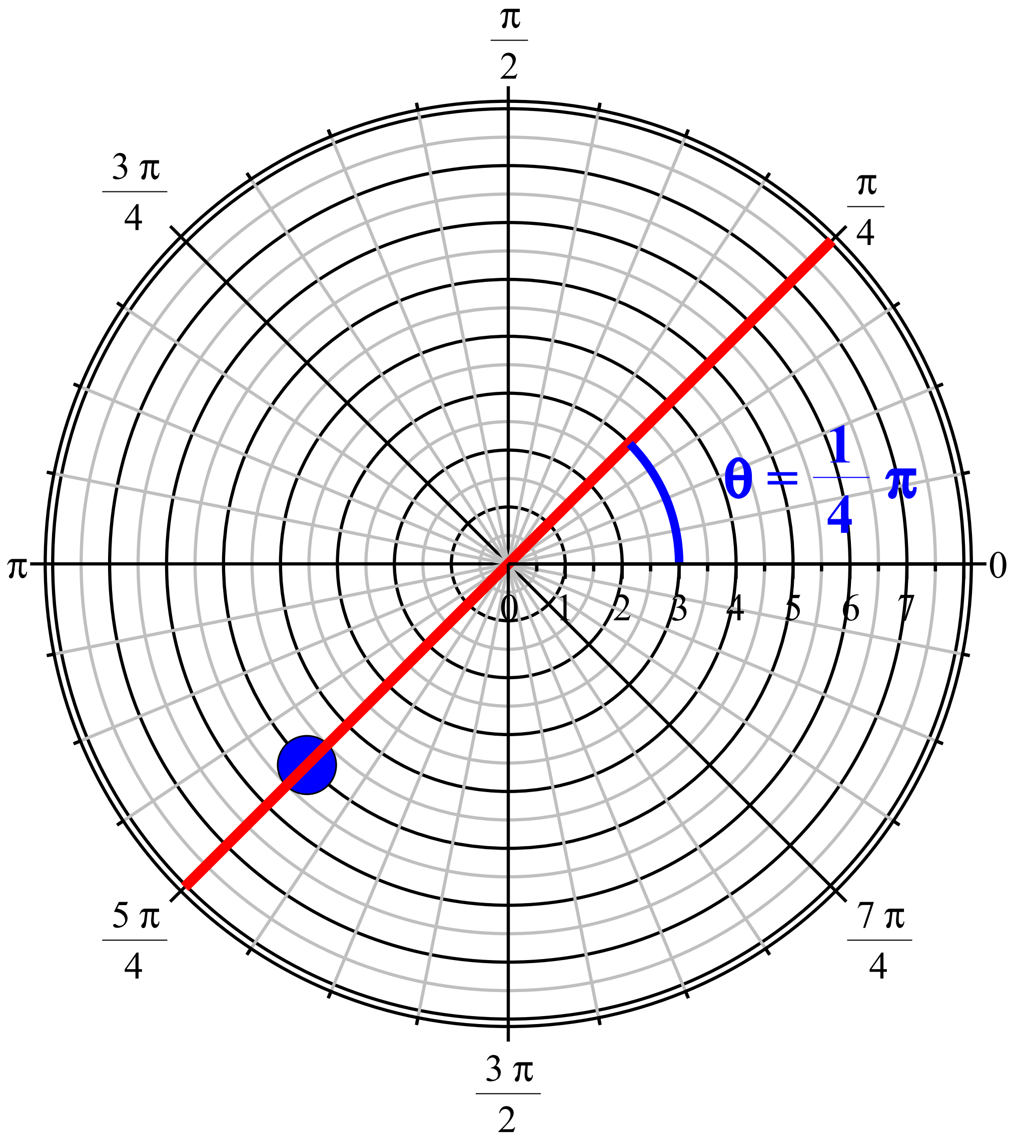


In general, an expression $r = a$ represents a circle of radius a centered at O.

Example: The simple expression $\theta = \frac{\pi}{3}$ is independent of r, so it represents the straight line passing through O that makes an angle of $\frac{\pi}{3}$ radians relative to the polar axis (the $+x$ axis).



Note: If the radial coordinate r is negative, *e.g.*, the point $(r, \theta) = \left(-5, \frac{\pi}{4}\right)$, then the point is along the ray $\theta = \frac{\pi}{4}$ but 5 units in the opposite direction.



Therefore, the polar representation of a point is not unique:

$$(r, \theta) = \left(-5, \frac{\pi}{4}\right) = \left(+5, \frac{5\pi}{4}\right) = \left(+5, -\frac{3\pi}{4}\right) = \left(-5, \frac{9\pi}{4}\right) = \cdot\ \cdot\ \cdot$$

Example: What is the polar representation of the circle of radius a centered at the point $(x, y) = (a, 0)$?

$$(x - a)^2 + y^2 = a^2$$

$$\Rightarrow \qquad x^2 - 2\,a\,x + a^2 + y^2 \;=\; a^2$$

where $x \;=\; r\cos\theta$ and $y \;=\; r\sin\theta$. Substitute into the above:

$$r^2\cos^2\theta - 2\,a\,r\cos\theta + r^2\sin^2\theta \;=\; 0$$

$$\Rightarrow \qquad r^2\left(\cos^2\theta + \sin^2\theta\right) \;=\; 2\,a\,r\cos\theta$$

$$\Rightarrow \qquad r \;=\; 2\,a\cos\theta\,.$$

So the polar representation of the circle of radius a centered at the point $(x, y) = (a, 0)$ is

$$\boldsymbol{r \;=\; 2a\cos\theta}, \qquad 0 \le \theta \le \pi\,.$$

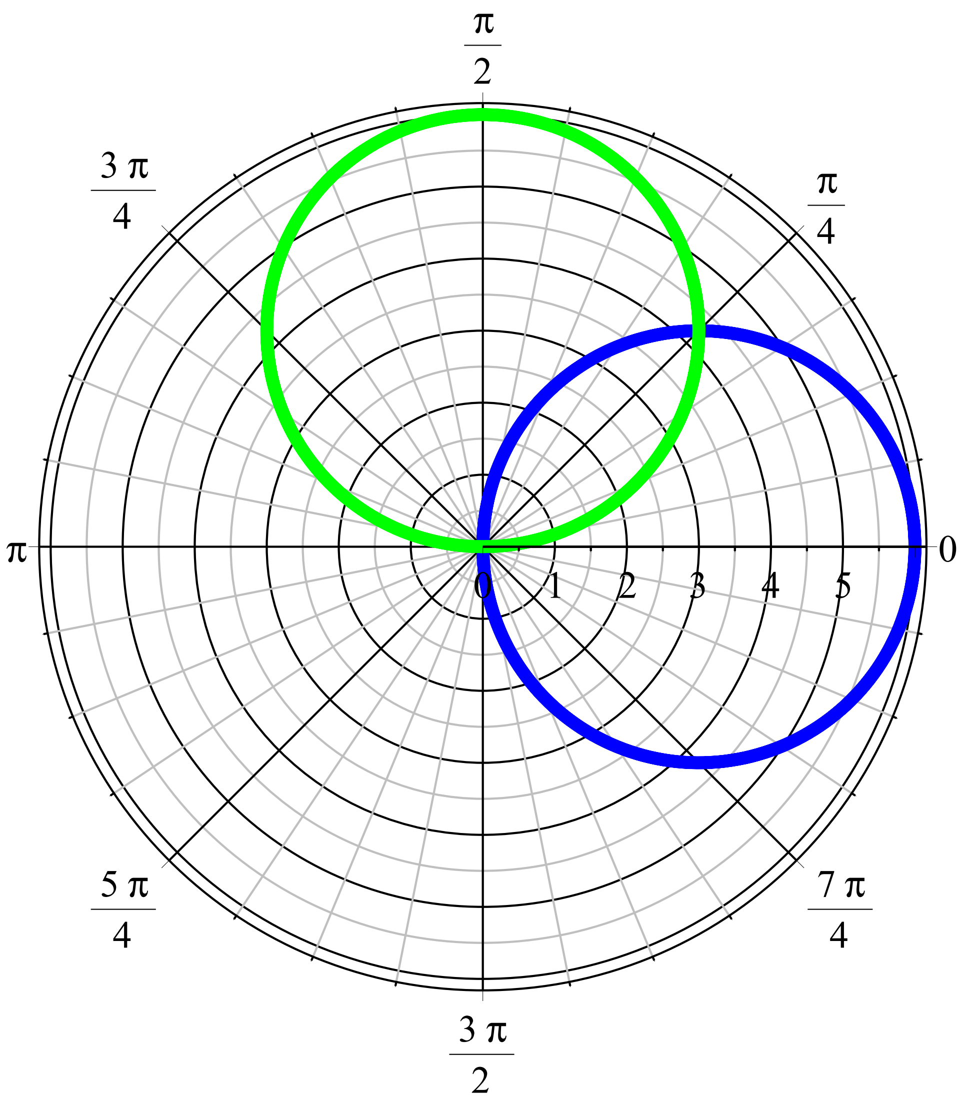
In a similar way, the polar representation of the circle of radius a centered at the point $(x, y) = (0, a)$ is

$$\boldsymbol{r \;=\; 2a\sin\theta}, \qquad 0 \le \theta \le \pi\,.$$

Example: The circle of radius 3 centered at the point $(x, y) = (3, 0)$ is (blue), and the circle of radius 3 centered at the point $(x, y) = (0, 3)$ is (green),

$$\boldsymbol{r \;=\; 6\cos\theta}, \qquad \boldsymbol{r \;=\; 6\sin\theta}\,.$$

```
> restart ;
> with(plots):  with(plottools):
> xcircle:=polarplot(2*3*cos(theta),axesfont=[times,roman,14],
  thickness=6, color=blue):
  ycircle:=polarplot(2*3*sin(theta),thickness=6, color=green):
  display([xcircle,ycircle],coordinateview=[0..6.1,0..2*Pi]);
```

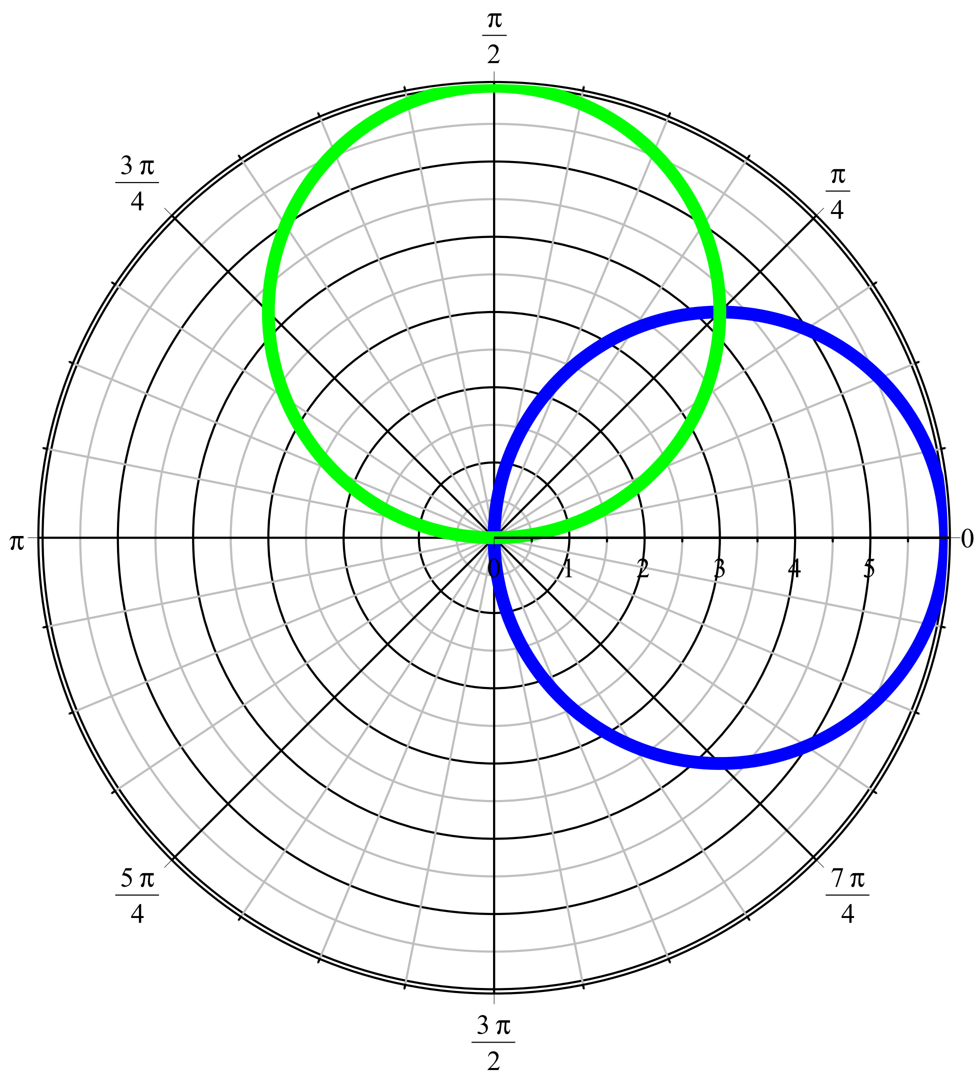


$r = 6 \cos \theta$, $r = 6 \sin \theta$

```
> xanim:=animatecurve( [6*cos(theta), theta,theta=0..Pi], coords=
  polar,color=blue,axiscoordinates=polar,frames=50,thickness=6,
  numpoints=200,scaling= constrained):

  yanim:=animatecurve( [6*sin(theta), theta,theta=0..Pi], coords=
  polar,color=green,axiscoordinates=polar,frames=50, thickness=6,
  numpoints=200,scaling=constrained):
```

```
display([xanim, yanim]);
```



$$r = 6\cos\theta, \qquad r = 6\sin\theta$$

Notice that each circle makes one revolution counterclockwise as θ varies from 0 to π.

Example: What do the equations represent?

$a)$ $r = -6\cos\theta$, b) $r = -6\sin\theta$.

They are:

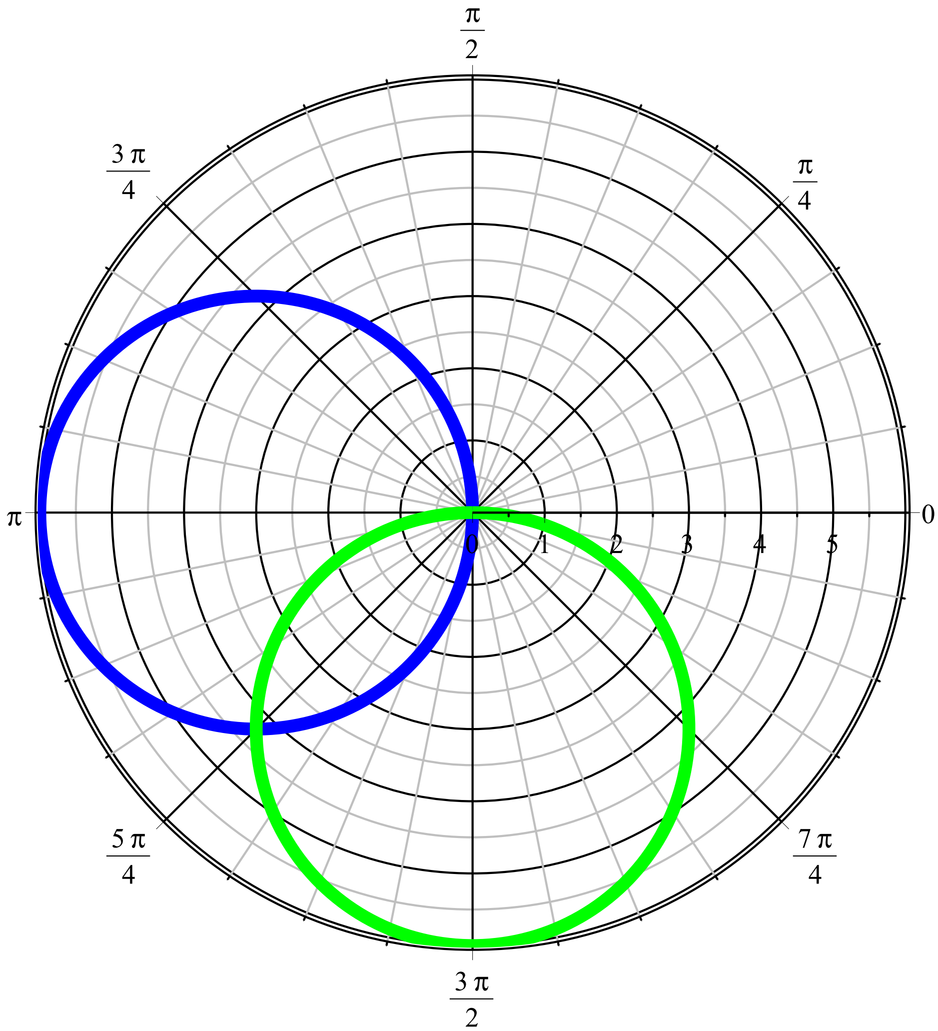
a) a circle of radius 3 centered at $(x, y) = (-3, 0)$ and

b) a circle of radius 3 centered at $(x, y) = (0, -3)$.

```
> xanim:=animatecurve( [-6*cos(theta), theta,theta=0..Pi],coords=
polar,color=blue,axiscoordinates=polar,frames=50,thickness=6,
numpoints=200,scaling=constrained ):

yanim:=animatecurve( [-6*sin(theta), theta,theta=0..Pi],coords=
polar,color=green,axiscoordinates=polar,frames=50,thickness=6,
numpoints=200,scaling=constrained ):

display([xanim, yanim]);
```



$$r = -6\cos\theta, \qquad r = -6\sin\theta.$$

Symmetry in Polar Coordinates

1. If a polar equation is unchanged when θ is replaced by $-\theta$, then the curve is symmetric about the polar axis (x axis). See the example $r = 6\cos\theta$: since

$$r = 6\cos(-\theta) = 6\cos\theta.$$

2. If a polar equation is unchanged when r is replaced by $-r$ OR θ is replaced by $\theta + \pi$, then the curve is symmetric about the pole (origin). This means the curve remains unchanged if it is rotated 180° about the origin.

3. If a polar equation is unchanged when θ is replaced by $\pi - \theta$, then the curve is symmetric about the line $\theta = \frac{\pi}{2}$ (y axis). See the example $r = 6\sin\theta$: since

$$r = 6\sin(\pi - \theta) = 6\sin(-\theta + \pi) = -6\sin(-\theta) = 6\sin(\theta).$$

Limaçons

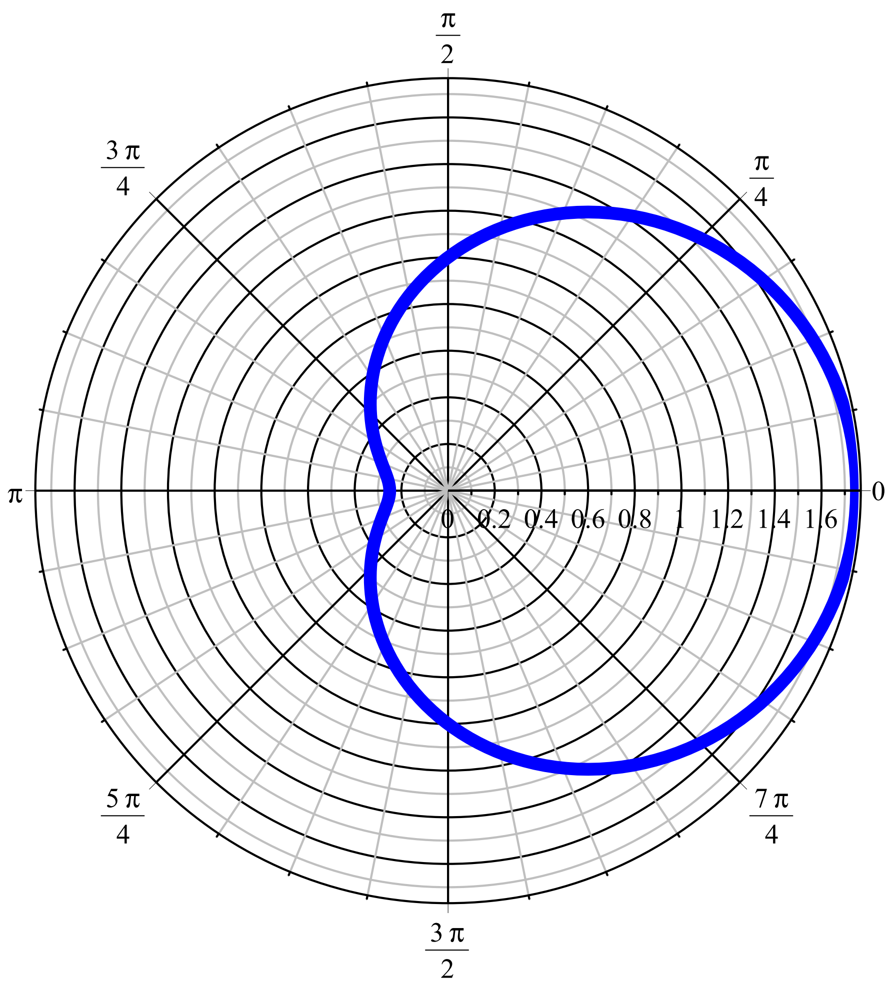
Consider a polar equation of the form

$$r = b + c\cos\theta\text{, for constants } b \text{ and } c.$$

For example, consider the limaçon

$$r = 1 + \frac{3}{4}\cos\theta.$$

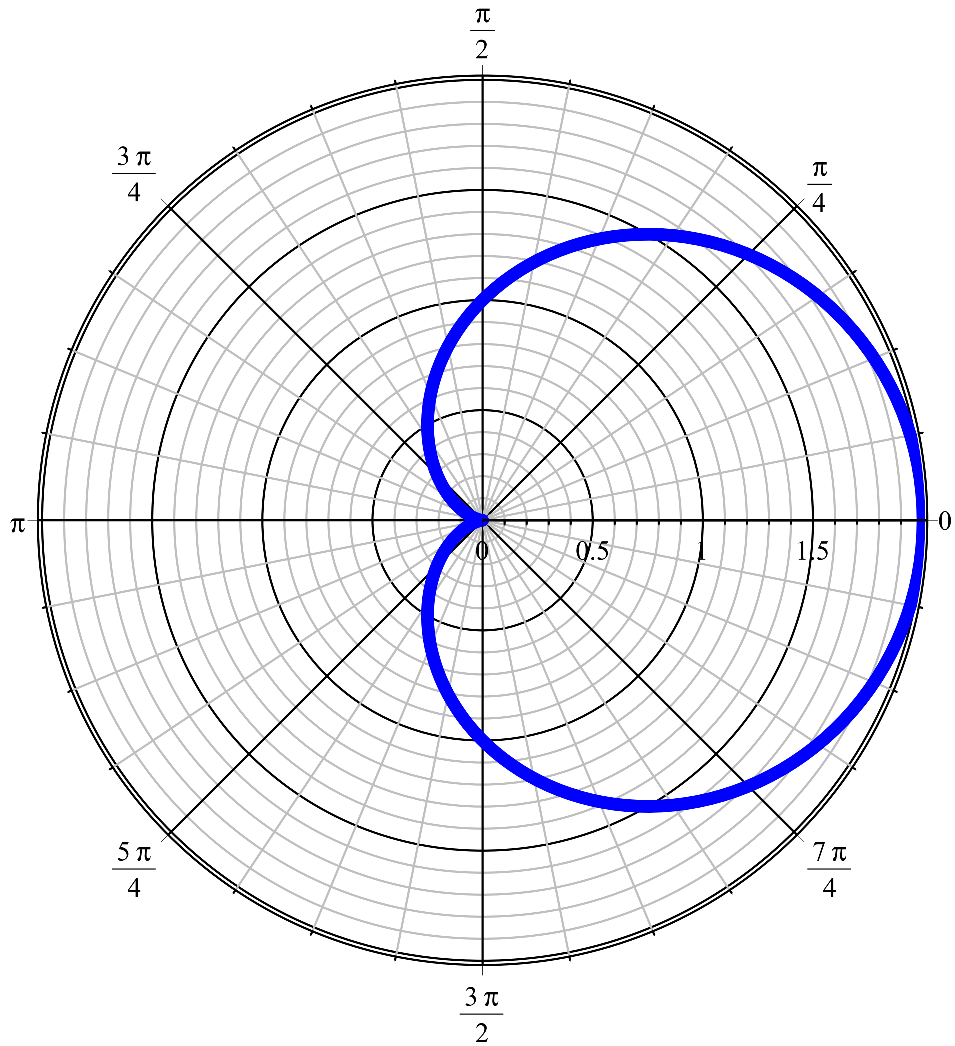
```
> animatecurve( [1+3/4*cos(theta), theta,theta = 0..2*Pi], coords=
  polar,color=blue,axiscoordinates=polar,frames=81, thickness=6,
  numpoints=200,scaling=constrained);
```



$$r = 1 + \frac{3}{4}\cos\theta$$

Now consider the limaçon: $r = 1 + \cos\theta$.

```
> animatecurve( [1+cos(theta), theta,theta = 0..2*Pi],coords=polar,
  color=blue,axiscoordinates=polar,frames=81, thickness=6,
  numpoints=200,scaling=constrained);
```

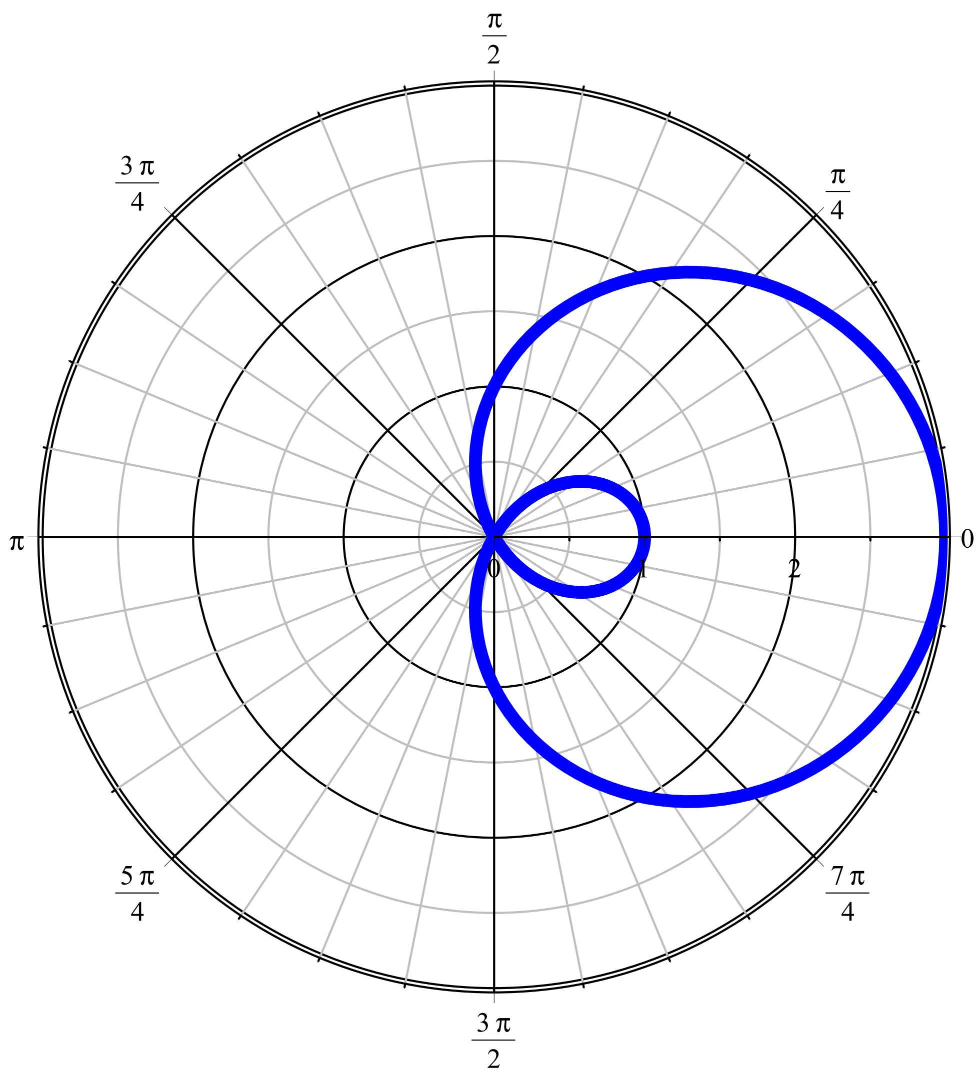


$$r = 1 + \cos\theta$$

This limaçon has a cusp at the pole. This curve is called a **cardioid**.

Now consider the limaçon: $r = 1 + 2\cos\theta$.

```
> animatecurve( [1+2*cos(theta), theta,theta = 0..2*Pi], coords=
  polar,color=blue,axiscoordinates=polar,frames=81, thickness=6,
  numpoints=200,scaling=constrained);
```



$$r = 1 + 2\cos\theta$$

This limaçon has a loop.

If fact,

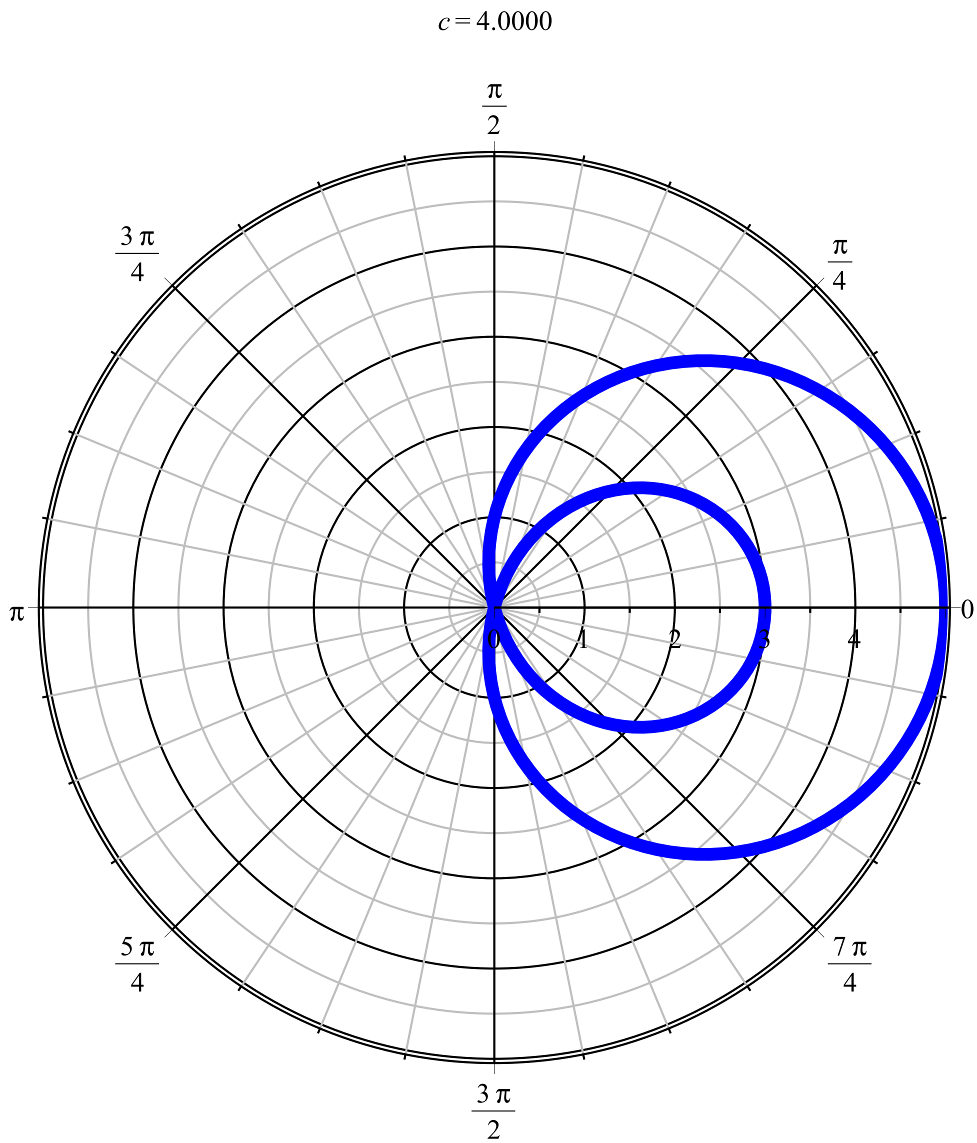
- if $c < 1$, the limaçon has a smooth dimple,
- if $c = 1$, the limaçon has a cusp (and the curve is called a **cardioid**),
- if $c > 1$, the limaçon crosses itself and has a loop.

Now let's consider

$$r = 1 + c \cos \theta$$

as we increase the c value of from $c = 0$ to $c = 4$.

```
> animate(plot, [1+c*cos(theta), theta = 0 .. 2*Pi,coords=polar,
  axiscoordinates=polar, color=blue, thickness=6, numpoints=200,
  scaling = constrained], c=0..4,frames=81,filled=[color="Blue",
  transparency=0.5] );
```



$$r = 1 + c\cos\theta$$

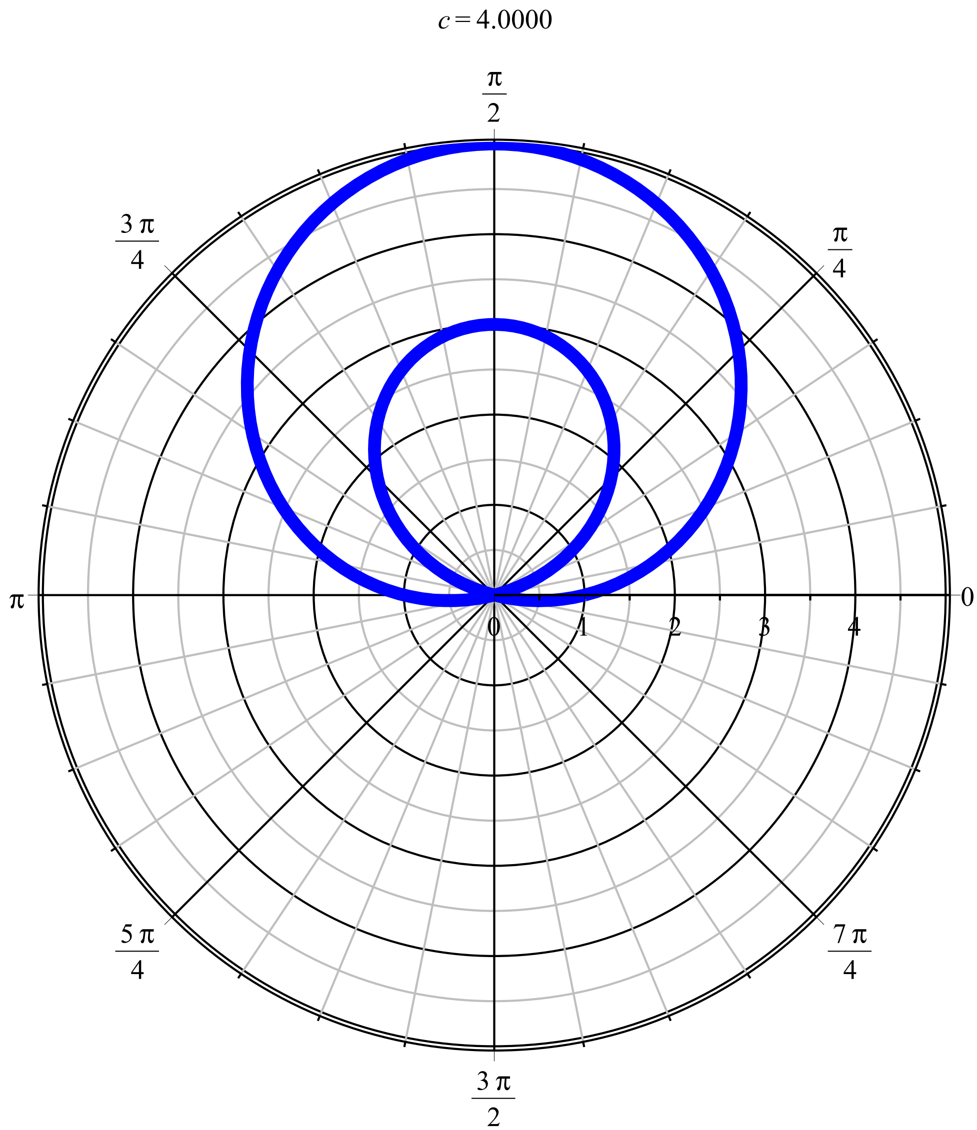
Likewise, consider

$$r = 1 + c\sin\theta$$

as we increase the c value of from $c = 0$ to $c = 4$.

```
> animate(plot, [1+c*sin(theta), theta = 0 .. 2*Pi,coords=polar,
```

```
axiscoordinates=polar, color=blue, thickness=6, numpoints=200,
scaling = constrained], c=0..4,frames=81 );
```



$$r = 1 + c \sin\theta$$

Roses

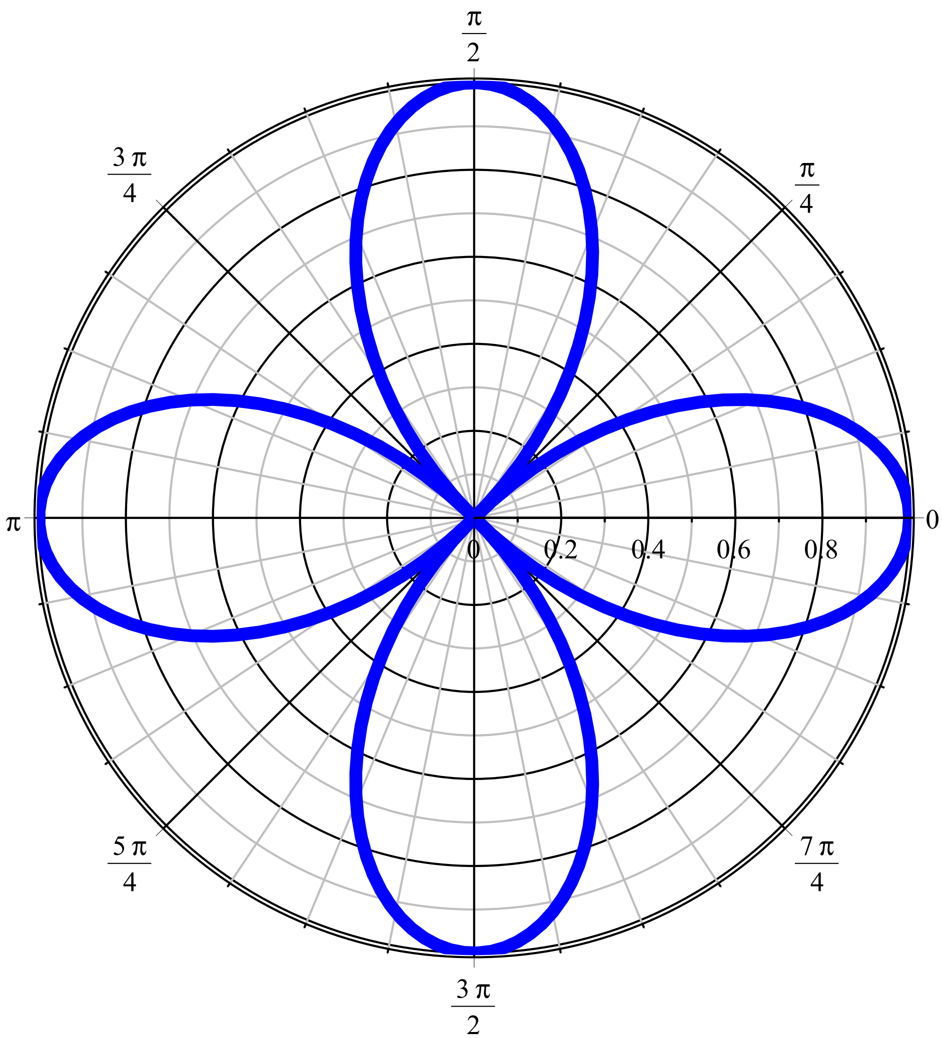
$r = \cos n\theta$ and $r = \sin n\theta$ where n is an integer.

- If n is even, then the rose has $\mathbf{2n}$ petals on interval $0 \leq \theta \leq 2\pi$.
- If n is odd, then the rose has $\boldsymbol{n}$ petals on interval $0 \leq \theta \leq \pi$.

Example:

Graph the rose $r = \cos 2\theta$. Since $n = 2$ (even), the rose as 4 petals on $0 \leq \theta \leq 2\pi$.

```
> animatecurve( [cos(2*theta), theta,theta = 0 .. 2*Pi], coords=
  polar,axiscoordinates=polar,frames=81, color=blue, thickness=6,
  numpoints=200, scaling = constrained );
```



$r = \cos 2\theta$

The top half of the 1st petal (on the $+x$ axis) occurs when $0 \leq \theta \leq \frac{\pi}{4}$.

The 2nd petal (on the $-y$ axis) occurs when $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$.

The 3rd petal (on the $-x$ axis) occurs when $\frac{3\pi}{2} \leq \theta \leq \frac{5\pi}{4}$.

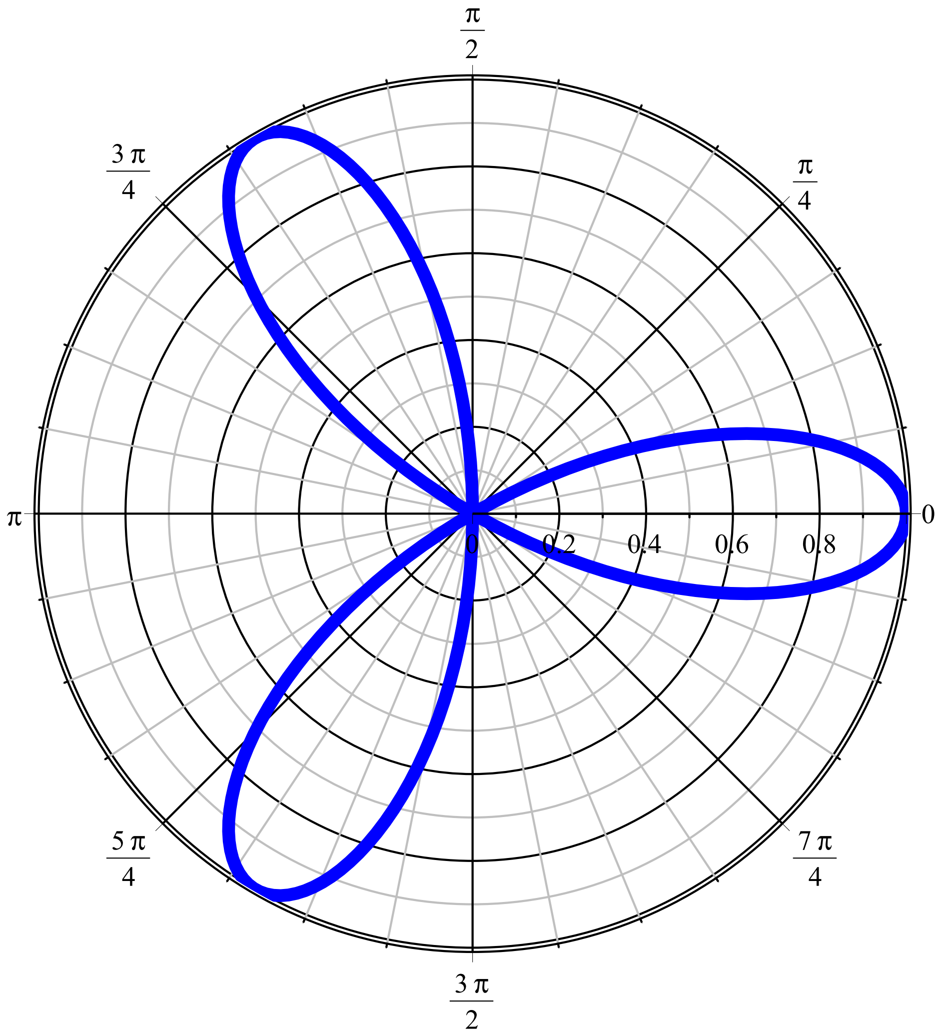
The 4th petal (on the $+y$ axis) occurs when $\frac{5\pi}{4} \leq \theta \leq \frac{7\pi}{4}$.

The bottom half of the 1st petal (on the $+x$ axis) occurs when $\frac{7\pi}{4} \leq \theta \leq 2\pi$.

Example:

Graph the rose $\boldsymbol{r = \cos 3\theta}$. Since $n = 3$ (odd), the rose as 3 petals on $0 \leq \theta \leq \pi$.

```
> animatecurve( [cos(3*theta), theta,theta = 0 .. Pi], coords=
  polar,axiscoordinates=polar,frames=81, color=blue, thickness=6,
  numpoints=200, scaling = constrained );
```



$$r = \cos 3\theta$$

The top half of the 1st petal (on the $+x$ axis) occurs when $0 \le \theta \le \frac{\pi}{6}$.

The 2nd petal (in Quad III along $\theta = \frac{2\pi}{6} = \frac{\pi}{3}$) occurs when $\frac{\pi}{6} \le \theta \le \frac{3\pi}{6}$.

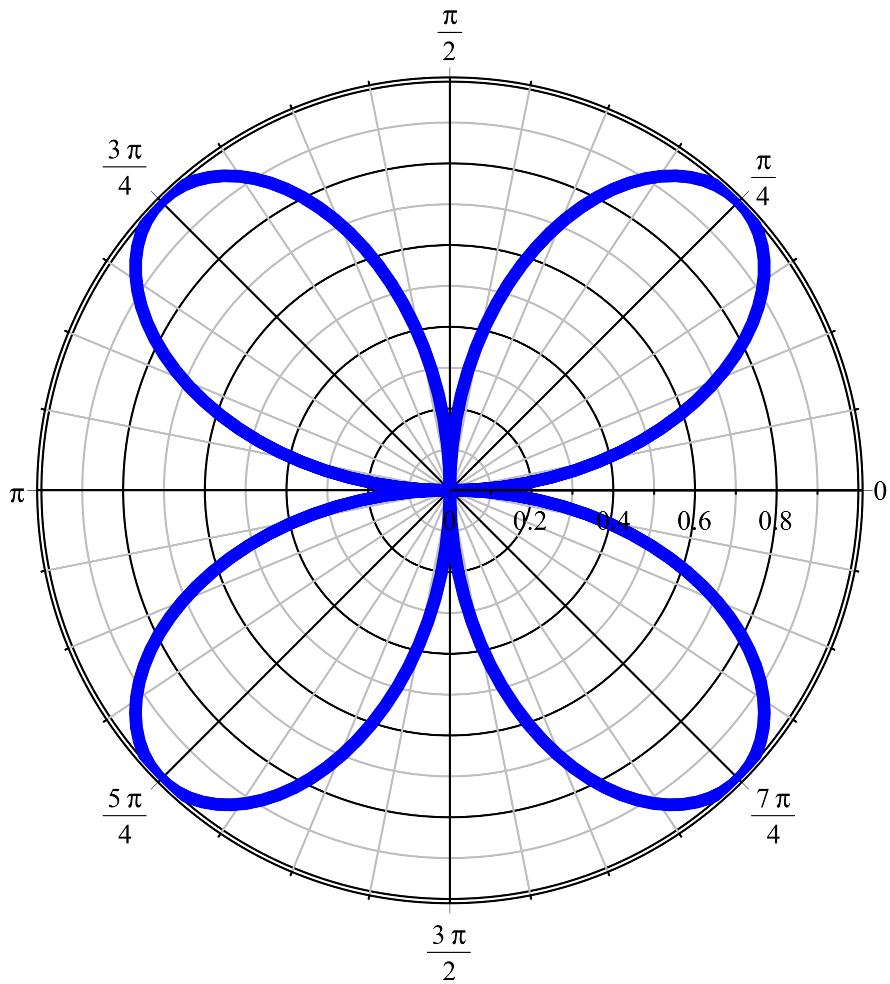
The 3rd petal (In Quad II along $\theta = \frac{4\pi}{6} = \frac{2\pi}{3}$) occurs when $\frac{3\pi}{6} \le \theta \le \frac{5\pi}{6}$.

The bottom half of the 1st petal (on the $+x$ axis) occurs when $\frac{5\pi}{6} \le \theta \le \pi$.

Example:

Graph the rose $\boldsymbol{r = \sin 2\theta}$. Since $n = 2$ (even), the rose as 4 petals on $0 \le \theta \le 2\pi$.

```
> animatecurve( [sin(2*theta), theta,theta = 0..2*Pi], coords=
  polar,color=blue,axiscoordinates=polar,frames=81,thickness=6,
  numpoints=200,scaling=constrained);
```



$\boldsymbol{r = \sin 2\,\theta}$

The 1st petal (in Quad I) occurs when $0 \le \theta \le \frac{\pi}{2}$.

The 2nd petal (in Quad IV) occurs when $\frac{\pi}{2} \le \theta \le \pi$.

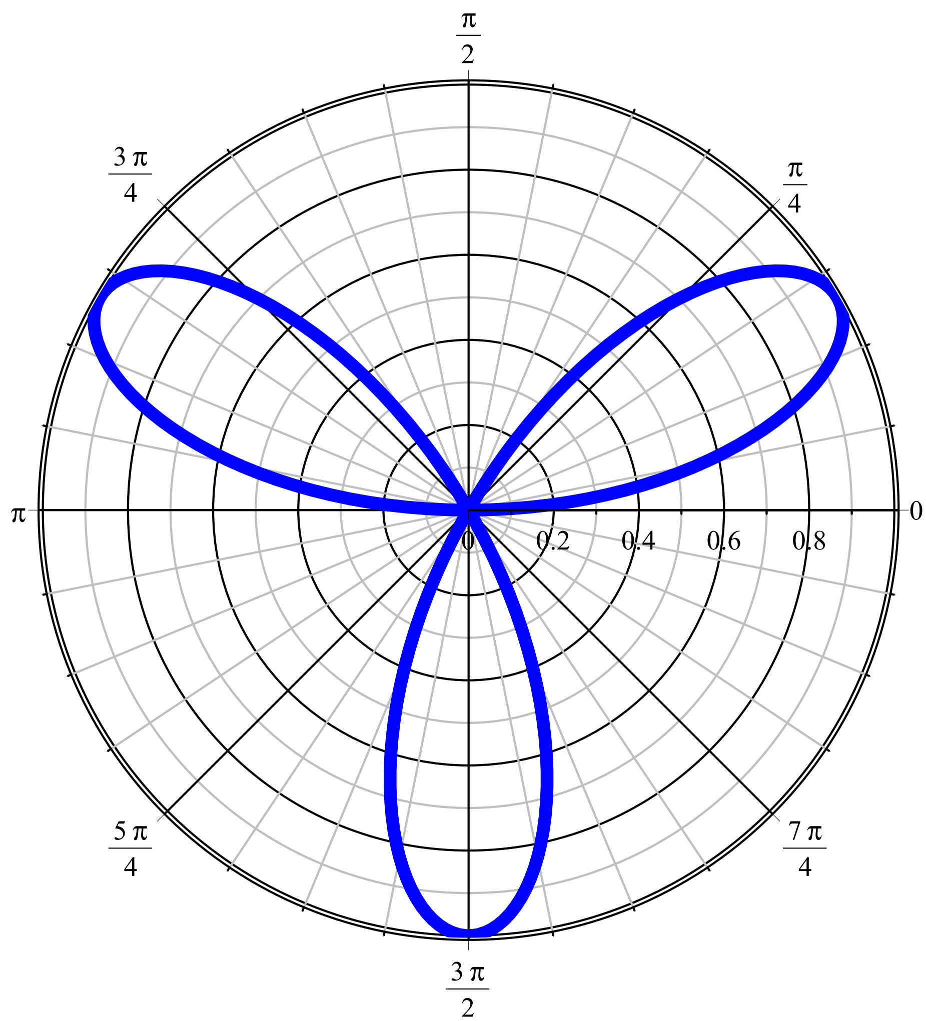
The 3rd petal (in Quad III) occurs when $\pi \leq \theta \leq \frac{3\pi}{2}$.

The 4th petal (in Quad II) occurs when $\frac{3\pi}{2} \leq \theta \leq 2\pi$.

Example:

Graph the rose $\boldsymbol{r = \sin 3\theta}$. Since $n = 3$ (odd), the rose as 3 petals on $0 \leq \theta \leq \pi$.

```
> animatecurve( [sin(3*theta), theta,theta = 0..Pi],coords=polar,
  color=blue,axiscoordinates=polar,frames=81,thickness=6,numpoints=
  200,scaling=constrained);
```



$$r = \sin 3\theta$$

The 1st petal (in Quad I along $\theta = \frac{\pi}{6}$) occurs when $0 \leq \theta \leq \frac{\pi}{3}$.

The 2nd petal (along the $-y$ axis) occurs when $\frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3}$.

The 3rd petal (in Quad II along $\theta = \frac{5\pi}{6}$) occurs when $\frac{2\pi}{3} \leq \theta \leq \pi$.

Slope of a Tangent Line in Polar Coordinates

Suppose we have a curve in polar coordinates given in the form

$$r = f(\theta).$$

Recall that the slope of the tangent line to a curve is $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$, where recall

$$x = r\cos\theta = f(\theta)\cos\theta \qquad \text{and} \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

So

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta \quad \text{and} \quad \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta.$$

Therefore,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

Example: Determine where the cardioid has horixontal and vertical tangent lines.

$$r = 1 + \sin\theta = f(\theta), \qquad 0 \le \theta \le 2\pi.$$

Here $f(\theta) = 1 + \sin\theta$, so $f'(\theta) = \cos\theta$. So

$$\begin{aligned}\frac{dy}{dx} &= \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos^2\theta - (1+\sin\theta)\sin\theta} \\ &= \frac{\cos\theta\,(1 + 2\sin\theta)}{1 - \sin^2\theta - \sin\theta - \sin^2\theta} \\ &= \frac{\cos\theta\,(1 + 2\sin\theta)}{1 - \sin\theta - 2\sin^2\theta} \\ &= \frac{\cos\theta\,(1 + 2\sin\theta)}{(1+\sin\theta)(1 - 2\sin\theta)}.\end{aligned}$$

a) The numerator is 0 if $\cos\theta = 0$ or $\sin\theta = -\frac{1}{2}$:

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \quad \text{and} \quad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}.$$

So the cardioid has horizonal tangent lines at the points with polar coordinates

$$(r, \theta) = \left(2, \frac{\pi}{2}\right), \left(2, \frac{3\pi}{2}\right), \left(\frac{1}{2}, \frac{7\pi}{6}\right), \left(\frac{1}{2}, \frac{11\pi}{6}\right).$$

b) The denominator is 0 if $\sin\theta = -1$ or $\sin\theta = \frac{1}{2}$:

$$\theta = \frac{3\pi}{2}, \qquad \text{and} \qquad \theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

NOTE: Since both numerator and denominator are 0 at $\theta = \frac{3\pi}{2}$, we must use L'Hospital's rule to determine the slope of the tangent line at $\theta = \frac{3\pi}{2}$.

a) The numerator is 0 when:

$$\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}.$$

So the cardioid has horizonal tangent lines at the points with polar coordinates

$$(r, \theta) = \left(2, \frac{\pi}{2}\right), \left(\frac{1}{2}, \frac{7\pi}{6}\right), \left(\frac{1}{2}, \frac{11\pi}{6}\right).$$

b) The denominator is 0 when:

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

So the cardioid has vertical tangent lines at the points with polar coordinates

$$(r, \theta) = \left(\frac{3}{2}, \frac{\pi}{6}\right), \left(\frac{3}{2}, \frac{5\pi}{6}\right).$$

Now we use L'Hospital's rule to determine the slope of the tangent line at $\theta = \frac{3\pi}{2}$.

$$\lim_{\theta \to \frac{3\pi}{2}} \frac{dy}{dx} = \lim_{\theta \to \frac{3\pi}{2}} \frac{\cos\theta + 2\sin\theta\cos\theta}{1 - \sin\theta - 2\sin^2\theta} \quad \text{trig identity}: \quad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \lim_{\theta \to \frac{3\pi}{2}} \frac{\cos\theta + \sin 2\theta}{1 - \sin\theta - 2\sin^2\theta} \sim \frac{0}{0}$$

$$= \lim_{\theta \to \frac{3\pi}{2}} \frac{-\sin\theta + 2\cos 2\theta}{-\cos\theta - 4\sin\theta\cos\theta} \sim \frac{0}{0} \quad \text{by L'Hospital's rule}$$

$$= \lim_{\theta \to \frac{3\pi}{2}} \frac{-\sin\theta + 2\cos 2\theta}{-\cos\theta - 2\sin 2\theta} \quad \text{since} \quad \sin 2\theta = 2\sin\theta\cos\theta$$

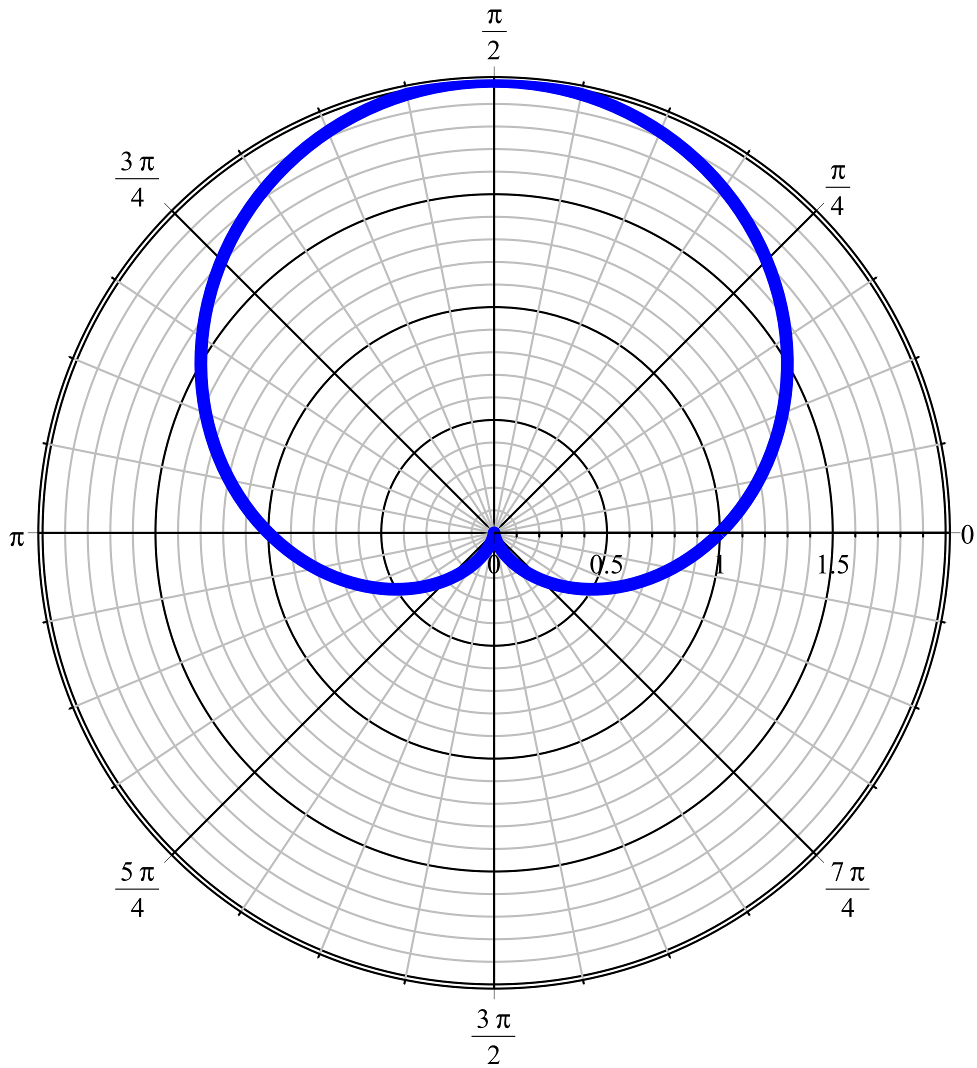
$$= \frac{-(-1) + 2\cdot 0}{0 - 2\cdot 0}$$

$$= \frac{1}{0} = \infty.$$

So the cardioid also has a vertical tangent line at the point

$$(r, \theta) = \left(0, \frac{3\pi}{2}\right) = (0, 0), \text{ since this is the origin (the pole).}$$

```
> plot(1+sin(theta), theta = 0..2*Pi,coords=polar,axiscoordinates=
  polar, color=blue, thickness=6,numpoints=200, scaling=
  constrained);
```



$\frac{\pi}{2}$
$\frac{3\pi}{4}$
$\frac{\pi}{4}$
π
0
0
0.5
1
1.5
$\frac{5\pi}{4}$
$\frac{7\pi}{4}$
$\frac{3\pi}{2}$