Recall: An eqn

$$F(x, y, z) = k$$

generally represents a surface in $\mathbb{R}^3$.

The tangent plane to that surface at pt (x_0, y_0, z_0) is given by

$$\nabla F(x_0, y_0, z_0) \cdot (\vec{r} - \vec{r}_0) = 0$$

written out:

$$F_x(x_0, y_0, z_0)(x - x_0)$$
$$+ F_y(x_0, y_0, z_0)(y - y_0)$$
$$+ F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Note: vector $\nabla F(x_0, y_0, z_0)$ is normal to the surface

$$F(x, y, z) = k$$

at pt (x_0, y_0, z_0).

Ex. Find the eqn of the tangent plane to the sphere

$$x^2 + y^2 + z^2 = 36 \qquad (1)$$

at $(2, 4, -4)$. So here,

$$F(x, y, z) = x^2 + y^2 + z^2$$

$$x_0 = 2, \quad y_0 = 4, \quad z_0 = -4$$

Then

$$F_x = 2x \Rightarrow F_x(2,4,-4) = 4$$

$$F_y = 2y \Rightarrow F_y(2,4,-4) = 8$$

$$F_z = 2z \Rightarrow F_z(2,4,-4) = -8$$

So the tangent plane to the sphere at $(2,4,-4)$ is

$$4(x - x_0) + 8(y - y_0) + -8(z - z_0) = 0$$

$$\Rightarrow \quad 4(x-2) + 8(y-4) - 8(z+4) = 0$$

$$\Rightarrow \quad x + 2y - 2z = 18$$

Q. What is the normal line to the surface at $(2,4,-4)$?

The normal vector we already found is parallel to the line:

$$\vec{n} = \nabla F(2,4,-4)$$

$$= \langle 4, 8, -8 \rangle$$

or simply take

$$\vec{n} = \langle 1, 2, -2 \rangle = \langle a, b, c \rangle$$

Then the normal line to the surface at $(2,4,-4)$ is

$$x = x_0 + at \Rightarrow x = 2 + t$$

$$y = y_0 + bt \Rightarrow y = 4 + 2t$$

$$z = z_0 + ct \Rightarrow z = -4 - 2t$$

or $$\frac{x-2}{1} = \frac{y-4}{2} = \frac{z+4}{-2}$$

Interpretation: If

$$w = F(x, y, z) \qquad (1)$$

is a ftn of 3 variables, then if we set $w = k$, then the eqn

$$F(x, y, z) = k \qquad (2)$$

is a level surface of $F(x, y, z)$.

If pt (x_0, y_0, z_0) is on the level surface (2), then the gradient

$$\nabla F(x_0, y_0, z_0)$$

a) is normal to the level surface (2),

b) pts in the direction in which ftn $F(w)$ increases most rapidly.