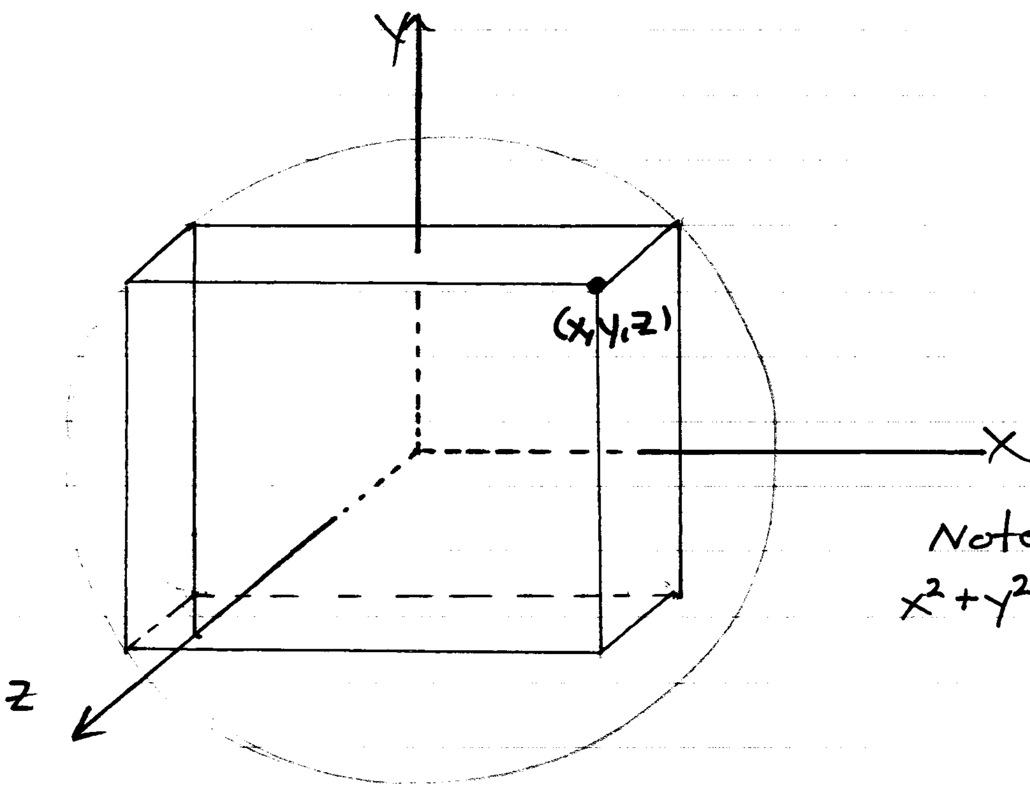
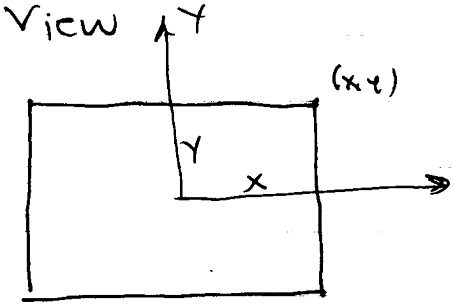
12.8

Ex 4 ~~Opt.~~ Find the dimension of the box of max. volume that can be inscribed in a sphere of radius 6.

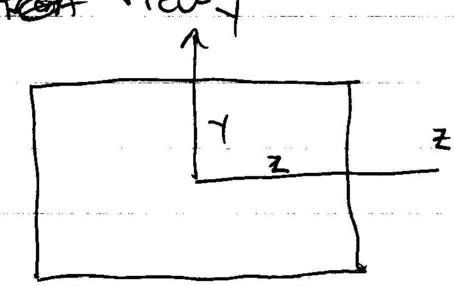


Note:
$x^2+y^2+z^2=36$

Side View



Back ~~Front~~ View



So

12-8

$$V = xyz \equiv f \qquad (M)$$

subject to

$$\underbrace{x^2+y^2+z^2}_{g} = 6^2 \qquad (C)$$

Since x, y, z represent lengths, we require $x>0,\ y>0,\ z>0$.

1. So

$$f_x = yz \qquad g_x = 2x$$
$$f_y = xz \qquad g_y = 2y$$
$$f_z = xy \qquad g_z = 2z$$

Set up

$$f_x = \lambda g_x$$
$$f_y = \lambda g_y$$
$$f_z = \lambda g_z$$
$$x^2+y^2+z^2 = 36$$

$\Rightarrow$

$$yz = 2\lambda x \qquad (1)$$
$$xz = 2\lambda y \qquad (2)$$
$$xy = 2\lambda z \qquad (3)$$
$$x^2+y^2+z^2 = 36 \qquad (4).$$

From (1-3):

$$2\lambda = \underbrace{\frac{yz}{x} = \frac{xz}{y}}_{(a)} = \frac{xy}{z} \qquad (5)$$

$$(b):\ \frac{yz}{x} = \frac{xy}{z} \qquad (c):\ \frac{xz}{y} = \frac{xy}{z}$$

From (a),

$$y^2 z = x^2 z$$

$$\Rightarrow$$

$$z(x^2 - y^2) = 0$$

$$\Rightarrow$$

$z = 0$ or $y = \pm x$. $\Rightarrow$ $y = x$ since $x > 0$ & $y > 0$.

But we don't want $z = 0$ since box would have no height.

So if

1. $y = +x$, then from (5),

$$2\lambda = z = \underbrace{z = \frac{x^2}{z}}$$

$$\Rightarrow \quad x^2 = z^2$$

$$\Rightarrow \quad x = \pm z \quad \Rightarrow \quad x = z$$ since $x > 0$ & $z > 0$.

So in this case

$$y = x \quad \& \quad z = x$$

so from (4)

$$x^2 + y^2 + z^2 = 36$$

$\Rightarrow$

$$3x^2 = 36$$
$$x^2 = 12$$

$$x = \pm 2\sqrt{3} \quad = +2\sqrt{3}$$ since x is a length.

$\Rightarrow$

$$x = y = z = 2\sqrt{3}$$

So pt is

$(2\sqrt{3}, 2\sqrt{3}, 2\sqrt{3})$ a CP subject to constraint (C).

Now

$$V(2\sqrt{3}, 2\sqrt{3}, 2\sqrt{3}) = \sqrt{12} \times \sqrt{12} \times \sqrt{12}$$
$$= 12\sqrt{12}$$
$$= 24\sqrt{3}$$

Other pts,

$$V(0,0,6) = 0$$
$$V(0,6,0) = 0$$
$$V(6,0,0) = 0.$$

So max volume is $24\sqrt{3}$ when dimensions are

$\sqrt{12} \times \sqrt{12} \times \sqrt{12}$.